RELATIONSHIPS BETWEEN COMMUNITY, INTERACTIONS, AND WAYS OF KNOWING IN COLLEGE PRECALCULUS CLASSES

By

JACQUELINE RENE COOMES

A dissertation submitted in partial fulfillment of
the requirements for the degree of

DOCTOR OF PHILOSOPHY

WASHINGTON STATE UNIVERSITY
College of Education

DECEMBER 2006

To the Faculty of Washington State University:

The members of the Committee appointed to examine the dissertation of JACQUELINE RENE COOMES find it satisfactory and recommend that it be accepted.

Chair

ACKNOWLEDGMENTS

I completed this project with the generous support of many people. I greatly appreciate Mr. Reilly and Mr. Anderson for opening their classes and sharing the wisdom gained from years of teaching. I learned a great deal about my own teaching from observing them. I thank the Chair of my committee, Amy Roth McDuffie, for her guidance. She offered support and challenges in amounts just right to keep me thinking and motivated. When I have a chance to mentor, she will be my model. I also thank the other members of my committee, David Slavit and Kim Vincent, for their many invaluable suggestions and insights. My colleagues in the Mathematics Department at EWU provided encouragement throughout the entire process, while my students continued to be a source of inspiration and insight.

My husband, Mark, has been my greatest supporter for more than twenty-six years, while our sons, Robert, David, and Anthony, have been my models of courage. Also, throughout the process, when I needed to take breaks in the work, my husband, sons, daughter-in-law Marina, and grandchildren, Natalie and Alex, provided the laughter, love, and support I needed to continue.

RELATIONSHIPS BETWEEN COMMUNITY, INTERACTIONS, AND WAYS OF KNOWING IN COLLEGE PRECALCULUS CLASSES

Abstract

by Jacqueline Rene Coomes, Ph.D.
Washington State University
December 2006

Chair: Amy Roth McDuffie

National standards for learning mathematics increasingly emphasize learning mathematics with understanding and students as agents of their own and others' understanding. However, students may have limited experiences as agents of their own and others' understanding, while classroom community and interactions may either foster or constrain roles and behaviors that could contribute to more connected, independent, and contextual knowing. This study examined the relationship between students' ways of knowing in college precalculus classes and factors related to classroom social norms, instructor and student roles, and interactions related to mathematical activity.

A qualitative two-case study approach was used to describe and analyze the development and nature of classroom community and interactions in two small community college classes during an eight-week summer quarter. Using Baxter Magolda's (1992) framework for development of students' ways of knowing in college, factors that were found to maintain students as absolute knowers included students' expectations of their roles, their entering perceptions of mathematics and learning mathematics, and the maintenance of instructor and student roles that placed mathematical and intermediate authority with the instructors. Factors that showed evidence of challenging students' absolute ways of knowing included instructors' explicit emphasis on understanding concepts, inclusion of real-life uses of mathematics, and portrayal of mathematics as uncertain. In addition, students' ways of knowing appeared to affect their perception of what they meant when they claimed a mathematical idea made sense.

The discussion includes a framework classifying evidence of students' ways of knowing and evidence of constraints and affordances in instruction. When instructors and students are at the same level, students' ways of knowing are maintained. However, when instruction is provided at levels higher than students evidence, it confirms and challenges students' ways of knowing. Suggestions for supporting and challenging students' growth as learners are included.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter One: Introduction

Influenced by constructivist theories of learning, national standards for teaching and learning mathematics emphasize the importance of fostering learning environments that encourage students' sense-making (American Mathematical Association of Two Year Colleges [AMATYC], 1995; National Council of Teachers of Mathematics [NCTM], 1991). In particular, students may develop dispositions to become agents of understanding mathematics and persevere in solving problems in classroom communities where their mathematical ideas are valued and opportunities they have to listen to and respond to the ideas of others (NCTM, 1991). The purpose of this research was to examine the development and nature of community and interactions in two small community college precalculus classes, to understand the factors related to community and interaction that supported or constrained students' roles in mathematical sense-making, and to understand how students' ways of knowing were related to these factors.

The first chapter of this dissertation lays the foundation for the study by providing background on students' ways of knowing mathematics and characteristics of community and classroom interaction that influence students' ways of knowing mathematics. I also discuss work experiences that aroused my interest in this topic. The second chapter reviews and synthesizes the literature related to learning mathematics, community, interactions, and students' ways of knowing. The third chapter details the methods used to answer the research questions while the fourth and fifth chapters explicate the findings of the two cases. The final chapter discusses the findings in light of the questions.

Ways of Knowing

Hiebert et al. (1997) described "sense-making," or learning mathematics with understanding as "… getting inside it and seeing how things work, how things are related to each other, and why they work like they do" (p. 2). The authors added that to learn with understanding students need to reflect on their experiences and communicate; reflection provides opportunities for making connections between ideas, while communication includes sharing their ideas and critically listening to others' ideas. By reflecting on and sharing their mathematical ideas, students become agents of their own and others' understanding.

However, some students may not know how to learn mathematics by sharing their ideas, listening to, and challenging others' ideas. Their assumptions about the nature and certainty of knowledge may encourage them to be received knowers, learners who believe the teacher has knowledge and students' roles are to gain it from them (Baxter Magolda, 1992; Belenky, Clinchy, Goldberger, and Tarule, 1997; Boaler & Greeno, 2000). To a received knower, the goal of learning is to "receive, retain, and return the words of authorities" (Belenky et al., p. 39). Consequently, they strive to memorize procedures without understanding.

Baxter Magolda (1992) developed a hierarchy of college students' ways of knowing, ranging from absolute knowers who believed all knowledge was certain and could be gained from authorities, to contextual knowers who believed knowledge was uncertain but that judgments could be made based on context. Contextual knowing is consistent with learning with understanding since students are expected to discuss and compare ideas, compare and evaluate solution processes, and authority rests with making sense rather than a declaration of right or wrong by the teacher.

Community

In their study of high school AP Calculus students in California, Boaler and Greeno (2000) found many students still experienced mathematics in traditional classrooms where the teacher was the only one who talked and students were not asked to make meaning or connections. Students in these classrooms believed that mathematics must be learned this way and avoided pursuing higher mathematics because it did not fit their identities as agents of their own understanding. In these didactic classroom communities, social norms afforded students only a passive role in learning, relationships between members of the class were not valued, and students believed that mathematics was a rigid set of procedures that must be learned by memorization. Their ways of knowing mathematics were as receivers of knowledge from authoritative sources such as the teacher and textbook (Boaler & Greeno).

In contrast, students in discussion-oriented classes were more likely to want to continue in mathematics because being agents of their own understanding fit their developing identities and they valued the relationships that formed in their classroom communities (Boaler & Greeno, 2000). The social norms and values of the discussion-oriented classes offered students active roles in learning mathematics, mathematical authority resided in their own sense-making, and students believed mathematics was supposed to make sense as they constructed understanding supported by interactions with others in their class.

However, Boaler and Greeno (2000) used only student interviews; they did not observe the classes to look for students in the didactic classes who explored and made connections independent of the teacher, understanding the mathematics through their own sense-making. Nor did they observe to determine if the teachers emphasized understanding concepts and making connections rather than memorization of procedures without understanding. Likewise, observations might have revealed students in the discussion-based classes who were dependent on received knowledge, writing down what others produced during group work and listening for what they would need to memorize during class discussions.

Interaction

The *Professional Standards for Teaching Mathematics* (NCTM, 1991) elaborates on both teacher and student roles in discourse. Specifically, teachers should ask questions and orchestrate classroom discussions so that students offer their thinking, clarify their ideas, make conjectures, and listen to others. The questions teachers ask should elicit and challenge student thinking (NCTM, 1991). While examining student learning in an inquiry-based classroom, Yackel and Cobb (1996) examined the nature of discourse and social norms and developed the concept of sociomathematical norms, a type of social norm that relates to the evaluation of mathematical activity. Sociomathematical norms influence the types of discourse by focusing discussion on mathematical sense-making.

The contrast between social norms and sociomathematical norms can be demonstrated in the way students provide an explanation. In any class students should be expected to explain their reasoning, a social norm. However, what constitutes an acceptable mathematical explanation is a sociomathematical norm (Yackel & Cobb, 1996). In the classroom studied by Yackel and Cobb, explanations had to describe actions on experientially real mathematical objects. For example, when one student explained how to add 12 and 13 by offering, "One plus one is two, and 3 plus 2 is 5" (p. 470), other students quickly challenged the procedural explanation and pointed out that the ones were tens and when they were added, totaled twenty. The mathematical objects in this case were the tens and ones.

In contrast, Kazemi and Stipek (2001) gave examples of elementary school teachers who accepted procedural explanations. To model what she expected, one of the teachers shared a student's explanation with the class, "First there were eight people and three brownies. We divided two brownies in fourths, and each person got ¼. And we divided the last one into eighths, and we gave each person a fourth and an eighth" (p. 69). The sociomathematical norms of this classroom did not create an expectation that students should provide reasons *why* they divided the brownies the way they did (Kazemi & Stipek).

Sociomathematical norms support communication that focuses students' attention on mathematical concepts and provides occasion for them to continue thinking (Cobb, Boufi, McClain, & Whitenack, 1997). Brendefur and Frykholm (2000) distinguished types of communication as either supported by social norms or supported by sociomathematical norms. They suggested that social norms support the teacher as mathematical authority and transmitter of understanding, and contrast this idea with sociomathematical norms that support students' agency as the source of their ways of knowing. Low levels of communication are characterized by teachers posing closed questions and students sharing solutions without deeper explorations, while high levels of communication are characterized by students listening to and exploring each other's ideas. High levels of communication modify and illuminate students' thinking (Brendefur & Frykholm).

Problem Statement

My university colleagues and I work to establish classrooms as communities where students explore, make conjectures, share their ideas, reflect on others' ideas, and are expected to become agents of their own understanding. However, I have heard comments from our students such as, "The teacher makes it too hard; it doesn't have to be that hard" and, "He doesn't have to turn everything into a question; sometimes he could just tell us." Another colleague related a story about a recent incident in class where, in response to a student's question, she asked the class, "Well, what do you think?" A student at the back firmly said, "What do YOU think? We want to know what YOU think." Clearly, not all college mathematics students value opportunities to become agents of their own understanding, to explore and persevere, or to reflect on other students' ideas.

In another example, I watched a videotape of a graduate instructor teaching a small intermediate algebra class. While the graduate instructor was a relatively inexperienced teacher, she had incorporated good questioning and group work into her lessons. During this lesson, several of her questions were posed to initiate a high level of communication, and the problems on the group work handout were also at a high level. She had excellent wait time and looked around the room attempting to make eye contact with all the students. However, in spite of the level of the questions, very few students interacted at the level intended. After each question and a period of silence, a student contributed explanations that would indicate sense-making and satisfy Yackel and Cobb's (1996) sociomathematical norm of what constitutes a good explanation. However, the same person answered each time and I believe he and a couple of other students were the only ones who benefited from the exchanges. When given the group work, all students except one pair worked alone while the instructor circulated and asked and answered questions. Most of the students' questions were procedural. It was clear some students were not striving to understand how the procedure connected to the concept, although the teacher tried to direct their attention and ask for meaning. On occasion, she finally gave them a procedure.

So, while teachers strive to create communities and interactions in which learners act as agents of their own and others' understanding, their efforts frustrate some students. To avoid student frustration, when students seem unable to understand, a teacher may compromise their intentions for conceptual learning so the students can successfully complete the work (Confrey, 1990). However, I have seen significant change in some students' approaches over a term and believe factors related to community and interaction can influence students' willingness to make sense.

Overview of the Study

In order to describe and understand components of classroom community such as social norms, roles, and interactions, meaning and context are important and thus, a qualitative two-case study design was employed (Merriam, 1998). In particular, an interpretivist perspective that coordinates interactionism with social constructivism was used to examine both interactions within the classroom and individual student responses (Cobb & Yackel, 1995). Interactionism was used to analyze the processes of joint meaning-making during classroom interaction, while social constructivism was used to analyze student interpretations (Cobb & Yackel).

The setting for the study consisted of two small precalculus classes in community colleges in a medium-sized city in the northwest United States. Since precalculus courses are the first college-level mathematics courses for most students majoring in science, mathematics, or technology, understanding and being able to apply the concepts will impact further success. Precalculus classes will offer more variety in students' responses than higher-level classes where students have successfully completed several mathematics classes. Since one goal of this study was to understand how community and interactions develop, instructors who seemed motivated to teach for understanding and who were reported to foster interactive classrooms were selected.

Research Questions

The research described and analyzed the nature of classroom community and interactions and the relationship of these factors to students' ways of knowing. I employed a two-case study design to explore the development and continued negotiation of these characteristics in the context of two small precalculus classes in community colleges. Specific questions guided the research:

1. In the classes under study, what is the nature of the classroom community and how does it develop?
2. In the classes under study, what is the nature of the instructors' and students' interactions related to mathematical activity?

Significance of the Research

Previous research presented evidence of the connection between community, interactions, and students' ways of knowing. Boaler and Greeno (2000) described didactic and discussion-based classroom communities, but did not observe the classes and did not describe how specific factors within the community develop, teacher purposes for fostering aspects of community, or students roles in its development. Specifically, Boaler and Greeno posited that the community affected students' ways of knowing, but did not explore whether students' current ways of knowing affected the community. Nor did they observe to discern the specific factors of community that affected students' ways of knowing. Similarly, researchers have explicated the negotiation of sociomathematical norms in college classrooms, but in courses at a higher level than introductory college mathematics (Rasmussen, Yackel, & King, 2003; Yackel & Rasmussen, 2002). Instructors may initiate norms and interactions intending to support students' sense-making, but college freshmen have had many years of experience learning mathematics and respond to opportunities to make sense of mathematics in a variety of ways. Some may continue to try to memorize procedures without making connections to concepts, while others seek to make connections. This study aids our understanding of how students respond to constraints and affordances to make sense of the mathematics they are learning.

The next chapter presents current theories of learning mathematics and the research literature related to learning precalculus, classroom communities, the nature of interactions, and students' ways of knowing. The third chapter details the methods used in the study.

Chapter Two: Literature Review

This chapter contains a discussion of current theories of learning mathematics and describes existing research concerning student learning of precalculus concepts. In addition, it presents literature on aspects of the classroom community such as social norms and roles, interactions, and ways of knowing.

Learning Mathematics

Current student-centered theories of learning mathematics generally fall into one of two categories, constructivist and sociocultural, although there are variations within each of these categories that differ in their emphases. Cobb (2000) distinguishes between the extremes of these two categories:

> A key issue that differentiates these positions is the relationship between individual psychological processes and classroom social processes. At one extreme, researchers who take a strong psychological perspective acknowledge the influence of social interaction, but treat it as a source of perturbations for otherwise autonomous conceptual development…At the other extreme, sociocultural approaches in the Vygotskian tradition tend to elevate social processes above psychological processes. For example, it is argued in some formulations that the qualities of students' mathematical thinking are *generated by* or *derived from* the organizational features of the social activities in which they participated [italics in original]. (van Oers, 1996, p. 309)

Although there are variations of constructivism, most tenets of constructivism are attributed to Piaget and accept that students actively construct their mathematical knowledge (Cobb, 1988). In addition, Piaget's constructs of assimilation and accommodation account for learning when new knowledge must be integrated into existing cognitive structures or when existing structures need to be reorganized or expanded before new knowledge is integrated (Herscovics, 1996). Ernest (1996) provided the following list of ideas valued by mathematics educators subscribing to constructivism:

1. Sensitivity toward and attentiveness to the learner's previous constructions.
2. "Diagnostic" teaching attempting to remedy learner errors and misconceptions, with perturbation and cognitive conflict techniques as part of this.
3. Attention to metacognition and strategic self-regulation by learners.
4. The use of multiple representations of mathematical concepts.
5. Awareness of goals for the learner, and the dichotomy between learner and teacher goals.
6. Awareness of the importance of social contexts. (p. 346)

In contrast, sociocultural theorists differ from constructivists in their beliefs about the nature of mathematics and of learning mathematics. Van Oers (1996) describes mathematics as existing in formal and informal practices, and learning mathematics as enculturation, or "pupils' and teachers' pursuit of making sense of mathematics as it is embodied in various practices in the surrounding world" (p. 91). Vygotsky, the originator of many current sociocultural theories, argued that conceptual development is influenced primarily by social interactions and semiotic mediation (Cobb et al., 1997). Students use symbols such as written and oral language to think and their thinking is both enabled and constrained by their culture and the available symbols (Forman, 1989). Forman (2003) characterized culture as the relationships, motives, beliefs, norms, goals and values of the members, and asserted that certain higher mental functions necessary for learning mathematics such as selective attention, voluntary memory, and logical reasoning, are social processes.

While discussing the differences between radical constructivism, social constructivism, and sociocultural theories of learning, Confrey (1995) suggested that an alternative incorporating the constructivism and sociocultural theories might be possible. Cobb and Yackel (1995) developed such a theory, a type of social constructivism they called the emergent theory. This theory coordinates rather than combines sociocultural theories and constructivist theories of learning by examining the social interaction in classroom communities, the mathematical development of the community, and students' individual mathematical constructions (Cobb & Yackel). I used the emergent perspective to guide the design of the current study.

Numerous mathematics educators have described the nature of students' mathematical knowledge. Skemp (1987) categorized mathematics learning as instrumental or relational. A student who knows what they are doing and why has relational knowledge, while a student who applies rules and procedures without understanding why has instrumental knowledge. Instrumental knowing is still a common form of knowing mathematics in the United States. For example, in their study of AP Calculus students in California, Boaler and Greeno (2000) reported, "The majority of students interviewed from the traditional classes reported that the goal of their learning activity was for them to memorize the different procedures they met" (p. 181).

Instruction may be designed to encourage relational knowing or instrumental knowing. Pesek and Kirshner (2000) compared a group of students who received relational only instruction (R-O) on area and perimeter with a group of students who received instrumental instruction before relational instruction (I-R). The R-O instruction was designed to encourage students to construct their own ways of finding area and perimeter, while the instrumental instruction focused on memorizing and applying the formulas. Although the I-R group received more time in instruction overall, the instrumental learning appeared to interfere with their ability to learn relationally. Specifically, none of the I-R group could correctly explain why the formulas for the area of a rectangle or triangle worked, while the majority of students interviewed from the R-O group were able to give correct reasons. When solving problems, students in the R-O group used conceptual and flexible methods while students from the I-R group relied on the formulas they had memorized. The authors concluded that memorized procedures interfered with the development of relational understanding, and that teaching for relational understanding only took less class time than teaching formulas and procedures, followed by understanding.

Instrumental teaching is the predominant form of teaching in the United States according to Smith (1996) who synthesized literature on teaching and learning mathematics and characterized teaching mathematics as demonstrating procedures and learning mathematics as practicing the steps shown by the teacher. In particular, he stated, "The answers to all mathematical problems are known and found in textbooks. Teachers who control and interpret texts are the intermediate authorities for students on mathematical truths" (p. 391).

In contrast, Confrey (1990) suggested that constructivism imposes a duty on the teacher to promote *powerful* and effective constructions that she characterized as:

1. A structure with a measure of internal consistency;
2. An integration across a variety of concepts;
3. A convergence among multiple forms and contexts of representation;
4. An ability to be reflected on and described;
5. An historic continuity;
6. Ties into various symbol systems;
7. An agreement with experts;
8. A potential to act as a tool for further constructions;

9. A guide for future actions; and

10. An ability to be justified and defended. (p. 111)

In addition, she added that students often use only the seventh characteristic, agreement with the experts, by asking their teachers to determine if their work is correct. After considering current theories of learning mathematics, since the classes under study were focused on precalculus topics, it was also important to consider research on students' learning of functions.

Learning Precalculus

The content included in the precalculus courses of this study included polynomial, rational, exponential, logarithmic, and trigonometric functions, systems of equations and matrix solutions, and graphs of polynomial functions. The students previously studied intermediate algebra, including simplifying and solving expressions and equations, and have had experience with functions and graphing. This section will review the research concerning the nature of students' understandings of the main topic of precalculus, functions.

Researchers have found intuitions to be an important factor in student's understanding of functions (Dreyfus & Eisenberg, 1983; Petitto, 1979). Leinhardt, Zaslavsky, and Stein (1990) describe intuitions as notions built through experiences before formal instruction. Dreyfus and Eisenberg studied college students' intuitions about linearity, differentiability, and periodicity. Their findings suggested that students visualized function information when it was given graphically, but not when it was given algebraically. Petitto examined student strategies for solving rational equations and found some students relied on intuition while others relied solely on formal algorithms regardless of which approach was more efficient. The results suggested that mathematics educators must distinguish between intuitive and formal thinking and that either type of thinking alone is inferior to their combined and coordinated use. Many of the students in

Petitto's study had no idea that their previous knowledge should inform their algebraic manipulations while others lacked the necessary understanding of the structure of the arithmetic operations they performed to be able to use them in algebraic operations.

Sfard (1992) characterized students' conceptions of functions as structural or operational. At an operational level, students perceive an algebraic entity only as a process and tend to be limited to procedural actions, while conceptions at a structural level allow students to perceive and act on an algebraic entity as an object. Operational conceptions precede structural conceptions, but once students hold a structural conception, they can perceive functions as processes and flexibly move between the conceptions as necessary. Sfard found that some students tended to develop pseudostructural conceptions, that is, identify a function as a formula without meaning anything else, or consider a graph and its equation without connecting the two representations.

Dubinsky and Harel (1992) examined the change in students' process conceptions of functions in a study involving undergraduates in a discrete mathematics class. The researchers presented students with relations in a variety of representations, asked them to determine if they were functions and to explain their answers. The researchers described some students' notions of functions as action conceptions, that is, they were limited to unconnected manipulations. Similarly, Carlson and Oehrtman (2005) interviewed over 40 precalculus students and those who consistently discussed functions as entities that accepted inputs and produced outputs successfully reasoned through a variety of function-related tasks while students who provided incorrect responses appeared to be applying memorized procedures.

Researchers have found several common difficulties students have with functions, some of which can be traced to concept images formed from early experiences with functions. A concept image is the set of all mental pictures and properties a person associates with the concept (Vinner, 1983). For example, one difficulty identified by researchers is that some students believe every function must be linear (Markovits, Eylon, & Bruckheimer, 1988). The researchers concluded that this concept image resulted from students experiences since many students' first encounter with functions include only linear functions. Students also assumed that functions must be manipulable, must consist of quantities, cannot be constant, and must be continuous (Dubinsky & Harel, 1992; Markovits et al.).

Vinner compared 10th and 11th grade students' concept definitions and concept images of functions. The purpose of the study was to find out whether students used concept images, concept definitions, or both when determining whether a given relation was a function. Some students recalled a definition that resembled the textbook definitions they had been given, but many recalled incorrect definitions. Other concept images imposed properties of regularity or symmetry on graphs of functions and imposed one-to-oneness on functions. In addition, some students believed functions must be determined by rules, especially formulas given algebraically. Confrey and Smith (1995) suggested students' correspondence image of functions, functions given as algebraic rules, results from the domination of this approach to functions in curricula. As an alternative, Confrey and Smith suggested an approach in which students examine how sequences of the independent and dependent variables relate and change. Referred to as covariation, students focus on changes in number sequences in a tabular representation of a function. For example, with exponential functions an additive sequence of the independent variable results in a multiplicative sequence of the dependent variable.

Slavit (1997) extended the object-oriented, correspondence, and covariance views of function, suggesting that the covariance and correspondence views were complementary and that students may acquire an object-oriented view of functions by understanding properties of particular families of functions. He investigated students' development of concept image of function based on the properties of the specific types of functions they had encountered. He posited that reification of function arises from property-noticing of one type of function to noticing whether any particular function has or does not have the property.

Zaslavsky (1997) studied cognitive obstacles of 10th and 11th grade high school students and found five common obstacles that could be traced to prior learning of linear functions and quadratic equations. The five obstacles were: 1) The graph of a quadratic function was limited only to the visible part that was actually drawn. Some graphs did not indicate the y-intercept, so students assumed it did not have one. Some students also believed the parabola had vertical asymptotes; 2) Students constructed a quadratic function given its zeros and assumed the leading coefficient could be arbitrarily chosen. For example, they indicated that a quadratic function defined by one rule was the same as that defined by a constant multiple of the rule. Students sometimes gave quadratic equations when asked for an example of a quadratic function (cf. Knuth, 2000); 3) Students found the slope between two points to determine the leading coefficient in the general form of a quadratic function, the same procedure they used for finding the leading coefficient of linear functions. They also believed three points on a parabola could be collinear; 4) Students rejected a function as a quadratic function if the linear or constant term were zero; and 5) They used only the x-coordinate to determine the vertex and as a result decided two parabolas had the same vertex when one was a vertical translation of the other. The obstacles did not depend on teacher, mathematics ability, school, or time elapsed since completing the chapter on quadratic functions.

After finding that early cohorts of students had limited concept images of functions, Schwarz and Hershkowitz (1999) designed instruction so that ninth-grade students spent more time investigating a larger variety of functions and representations of functions. The curriculum did not offer students a definition of functions but consisted of problem situations for which they constructed functions. In the inquiry-based classroom students had graphing calculators and multirepresentational software available and could choose the representations they wanted to use while solving the problems. Students exposed to the new curriculum demonstrated a better understanding of the global attributes of functions than previous cohorts did.

Other common problems with functions include students' lack of understanding of the connections between algebraic and graphical representations of functions (Knuth, 2000; Markovits et al., 1988). In Knuth's study of 284 college-prep high school students, the majority of students did not understand that a point is on the graph of a function if and only if it satisfies the corresponding equation. Students were largely unsuccessful if a translation task required them to use a graph to determine solutions to functions written in algebraic form. In contrast, researchers have found that technology can help students connect algebraic, numerical, and graphical forms of functions. Dugdale (1993) found that students who worked with function transformations in a graphical context better understood the connection between the algebraic and graphical representations of functions and found similar results for students using graphing calculators to graph trigonometric identities. In another case, Abramovich (2005) observed students in an inquiry-based trigonometry class. After students found four apparently different answers to a trigonometric equation, they were motivated to examine identities. Finding multiple solutions also prepared them to use technology to analyze their solutions to the equation and explore a generalized statement of the problem.

Esty (2005) reviewed applications of the concept of inverse functions and contrasted his results with the ways widely used precalculus texts, in sections devoted to inverse functions, addressed the concept. The textbooks each provided an algorithm for finding inverses algebraically, usually interchanging x and y, then solving for y. However, Esty determined this algorithm was not used in later applications or procedures involving inverse functions. Rather, future applications of the concept require understanding the idea of an inverse function as a function that reverses a process, or as a function that returns the original x's when the original y's are input. How students learn is also affected by the classroom environment. In the next section I describe research that examined the nature of classroom communities.

Classroom Community

Traditionally, mathematics has been taught as a set of procedures to be memorized. Such classrooms are often characterized as "chalk and talk" classrooms; while teachers talk, students interact very little with each other or the teacher (Boaler, 1997; Boaler & Greeno, 2000). Within the traditional communities described by Boaler and Greeno, students experienced mathematics as a set of procedures, disjointed from real life, and learning as "received knowing." Students were not expected to discover or connect mathematical concepts on their own because mathematical authority belonged to the teacher and textbook. As a result, students often chose methods of solving problems based on non-mathematical cues from the textbook or teacher.

Boaler and Greeno (2000) interviewed AP Calculus students in six California high schools and found that students in didactic classrooms believed the passive role afforded to them in the classrooms was the way mathematics classes were supposed to be. As a result, they did not strive to devise solutions and understandings beyond the procedures they were given. In many cases students decided they did not want to pursue mathematics further because it lacked creativity and opportunities for deeper thinking; their roles as passive receivers of knowledge did not fit with their identities as thinking agents.

In contrast, discussion-based classrooms are often characterized by small group and whole class discussions of open-ended problems as students solve problems and communicate their reasoning (Boaler, 1997). Angier and Povey (1999) used the metaphor of spaciousness to describe activities, social interactions, and mathematics within Angier's discussion-oriented secondary classroom. They described spacious mathematics as characterized by open-ended challenging problems that allowed students to devise their own solutions and discover connections. The social norms of discussing solutions with classmates and finding multiple representations contributed to the spaciousness of mathematics. In this environment, students developed the confidence to correct the teacher and doubt the book, allowing authority to reside in students' own understanding of the mathematics.

Similarly, in Boaler and Greeno's (2000) study, students in discussion-oriented classes reported playing an active role in discourse by offering their understandings and solutions, which were used to develop understanding within the class. Students became relational agents, accountable for contributing to each other's understanding. When teachers listened to and valued students' ways of thinking and used this discourse to develop mathematical reasons for correct solutions, authority rested in mathematical reasoning and not teacher endorsement (Boaler, 1997; NCTM, 1991).

However, asking questions and listening to students' answers is not sufficient for relinquishing authority. Peressini and Knuth (1998) described a teacher who facilitated whole class discussion after giving students a task, but facilitated discussion to have students understand *his* process since he believed there was only one way to solve the problem. Teachers have also been observed having students share solutions, but then not using those solutions to develop concepts (Stigler, Fernandez, & Yoshida, 1996). Stigler et al. concluded that teachers must value and use students' contributions to foster students' willingness to strive for sense-making. In order to plan effectively to use student thinking, teachers must anticipate student thinking while they are planning and be open to student solutions that are different from their own. Students must be given the time to think and support to continue thinking, and how students think depends on their ways of knowing.

Ways of Knowing

Boaler (1997) and Boaler and Greeno (2000) suggested that students who learned mathematics in discussion-oriented and traditional classrooms have different ways of knowing mathematics. Boaler's secondary mathematics students in open-approach classrooms described mathematics in the real world as being similar to school mathematics because they had to think and solve open-ended problems. In contrast, students in the traditional classrooms believed school mathematics was nothing like mathematics needed in the real world because they performed most of the procedures out of context. On examinations, traditionally taught students were not able to transfer their knowledge of procedures to situations that required interpretations. In contrast, students whose classroom practices included interpreting contexts were better able to interpret the situations on exams, understand mathematical relationships, and draw on appropriate procedures.

Baxter Magolda (1992) interviewed university students and developed a hierarchy of college students' ways of knowing, ranging from absolute knowers who believed all knowledge was certain and could be gained from authorities, to contextual knowers who believed knowledge was uncertain but that judgments could be made based on context. She built on the seminal work of Perry (1970) who used interviews of college men to develop a similar scheme, and on Belenky et al. (1986/1997) who interviewed women, although not necessarily college women. Baxter Magolda found parallel tendencies for men and women.

Baxter Magolda's (1992) highest level of knowing, contextual knowing, is consistent with current recommendations by mathematics educators: students are expected to discuss and compare ideas, compare and evaluate solution processes, tasks should provide opportunities to use concepts in context, and authority rests with making sense rather than a declaration of right or wrong by the teacher (NCTM, 2000).

Baxter Magolda (1992) included descriptions of several domains for each level of knowing, including role of the learner, role of peers, role of the instructor, evaluation, and the nature of knowledge. See Table 1 for descriptions of each level and corresponding roles of the learner, peers, and instructor. Baxter Magolda also found the majority of freshmen (68%) were absolute knowers, while 46% of sophomores, 11% of juniors, and 2% of seniors were absolute knowers. The percent of transitional knowers increased each year from 32% of freshmen to 80% of seniors. She found no independent knowers among freshmen and only 16% of seniors. There were very few contextual knowers in college: 1% of juniors and 2% of seniors, but the rate grew to 12% a year after college.

Baxter Magolda (1992) suggested instructors must first teach responsively to students' ways of knowing before they can help students develop more complex ways of knowing, but in order to advance students' ways of knowing, instructors must contradict students' ways of knowing. In fact, she gave an example of a young man who enjoyed learning from his mistakes in class but who noticed that the women in the class did not speak up. When he asked them why, they told him how they disliked learning in that way. Baxter Magolda used this as evidence that teachers must balance validating students' current ways of knowing with providing contradictions to their ways of knowing. That is, by demanding students adapt to a new way of knowing, teachers may lose opportunities to help students develop more complex ways of knowing.

Baxter Magolda (1992) also introduced three principles to help students develop more complex ways of knowing: validate the student as a knower, situate learning in the students' own experience, and define learning as jointly constructed meaning. The first principle, validate the student as a knower, implies that teachers must elicit and value students' ideas so they begin to value their own ideas. It also implies that teachers must resist being an authority so that students can continue reflecting on their own ideas and the ideas of their classmates. In order to develop more complex ways of knowing, students must be willing to listen to other students' refutations of their ideas and reflect and make judgments. However, teachers must still be willing to use their expertise and authority to encourage students to reflect when it appears students are not questioning their own and their peers' ideas deeply enough (Baxter Magolda).

The second principle is situating learning in the students' own experience (Baxter Magolda, 1992). As part of this principle, tasks and examples should relate to students' experiences whenever possible. This also implies that students should have opportunities to engage in mathematics during class and to discuss their experiences of doing mathematics. When students discuss how they thought about a problem and the false directions they took in a solution path rather than just relating a procedure, the experience becomes part of their story. Students' reflections on their own stories may aid the development of a sense of authority (Baxter Magolda). This development of authorship must include an audience who listens and values their ideas.

However, Baxter Magolda (1992) mentioned that the academic content did not necessarily need to connect to students' experiences. She supported this with the story of a teacher who came early to class to talk to students, suggesting that teacher roles such as being supportive and approachable were important in validating and advancing students' ways of knowing. Specifically, when teachers evidenced caring attitudes and showed a willingness to be helpful to their students by answering questions, students tended to develop as knowers. "This caring attitude offers confirmation of students and their ideas and gives them a chance to know the professor better. Increasing recognition of the teachers' humanness contradicts the notion that they and their knowledge are beyond the reach of students" (p. 274). Finally, Baxter Magolda suggested teachers provide opportunities for students to increasingly see knowledge as uncertain.

Brew (2001) integrated the work of Baxter Magolda (1992) and Belenky et al. (1986/1997), whose framework was also used by Boaler and Greeno (2000). Brew used the developmental levels and domains provided by Baxter Magolda to study ways of knowing of women who had recently returned to school, specifically those in developmental mathematics classes. However, she also included another developmental level below absolute knowers, silence, from Belenky et al. Students exhibiting silence were often self-deprecating and did not believe understanding was important or possible. The difference between absolute knowing and silence is the extent to which students exhibiting silence depended on external authorities and had no voice of their own (Brew).

In summary, the affordances and constraints of classroom communities affect students' beliefs about the nature of mathematics, students' ways of knowing and identities as mathematics learners. While learning mathematics, students are learning about themselves: "both disciplinary knowledge and knowledge of the self co-evolve, and reflexively impact each other in their development" (Middleton, Lesh, & Heger, 2003, p.406). Teachers influence students' ways of knowing mathematics by creating spacious classroom communities in which they relinquish mathematical and intermediate authority and foster students' ability to be active agents of their own and each others' learning. While examining the nature of the community is important, it is paramount that the mathematical focus of the discussions and activities are considered, the subject of the next section.

Interaction Related to Mathematics

Sociomathematical Norms

Within spacious classroom communities, social norms, beliefs, and values help create and sustain types of interaction that promote conceptual and relational understanding; sociomathematical norms are classroom social norms that relate to evaluation of mathematical activity (Yackel & Cobb, 1996) and support student construction of mathematical understanding (Cobb et al., 1997). Because teachers are representatives of the larger mathematical community, their expectations are important in establishing and maintaining sociomathematical norms (Fraivillig, Murphy, & Fuson, 1999; Kazemi & Stipek, 2001; Yackel & Cobb, 1996). However, student beliefs and sociomathematical norms mutually influence each other (Kazemi & Stipek; Yackel & Cobb; Yackel & Rasmussen, 2002).

Teachers may initiate a sociomathematical norm by providing feedback to students concerning the nature of an acceptable explanation (Kazemi & Stipek, 2001; Lampert, 1990; Yackel & Cobb, 1996). For example, in Kazemi and Stipek's study of four elementary classrooms, teachers in high press classrooms pressed students to justify their solutions using mathematical concepts, while teachers in low press classrooms accepted solutions consisting of procedures. Sociomathematical norms in the second-grade classroom observed by Yackel and

Cobb required student explanations concerning actions on mathematical objects to relate to actions on representations of the mathematical objects. In another case, researchers found that students negotiated sociomathematical norms indicating acceptable justifications independent of the teacher. In Hershkowitz and Schwarz's (1999) study of a middle school classroom, after two students entered formulas in a spreadsheet and found an answer recursively, they decided that a solution found inductively needed further justification.

Simon and Blume (1996) studied the characteristics of a mathematical justification and developed a hierarchy of responses. In a task involving the area of a rectangle, preservice elementary education students justified the use of a formula by stating the formula was given by previous teachers. The instructor pressed students to justify mathematically and some students justified the formula for a particular case. When more sophisticated justifications were given, many students in the class were not able to distinguish how they were different and why some justifications were better than others. From this interaction, Simon and Blume created a hierarchy consisting of five levels: no justification at all, justification by authority of previous teachers or texts, demonstrations of particular cases, deductive reasoning in terms of specific cases, and deductive reasoning independent of specific cases.

In an extension problem, the teacher asked students to make a conjecture about the area of the interior of a closed irregular shape. A student suggested they could use a string to measure the perimeter and then form the string into a rectangle; the rectangle would have the same area as the irregular shape. When another student gave a counterexample demonstrating that this method does not work in general, the other students did not give up the strategy. They did not understand that this counterexample refuted their hypothesis. The researchers concluded that it is very difficult to change students' strongly held beliefs about how mathematics is validated (Simon & Blume, 1996).

Rasmussen, Yackel, and King (2003) contrasted acceptable explanations in two college differential equations classes. In one class, the teacher responded to a student's question of how he found an answer by restating the rule; this justification was sufficient. In the other class, the teacher probed further when students offered procedural explanations, cultivating a sociomathematical norm that student explanations include connections to rates of change. For example, while investigating an equation that modeled the rate of change of a squirrel population, the teacher responded to a student answer with, "Tell us why you made that conclusion" (p. 152). The student responded with a procedural answer and the teacher continued to press for meaning, "and so what does that mean for us?" (p. 152). Responses from another student indicated students listened to each other's explanations and the teacher's responses, and contributed to the development of this sociomathematical norm.

Similarly, Kazemi and Stipek (2001) observed teachers pressing students to make connections between mathematical concepts. For example, a high-press teacher in one classroom focused students' attention on the mathematical differences between solutions to help students compare relationships among the strategies. In contrast, a teacher in a low-press classroom called on various students until one gave her the answer she was seeking. When a student pointed out that the accepted answer was equivalent to the one he gave, the teacher responded that his answer was not really what she was looking for. By asking all students to explore whether the two solutions were the same, the teacher could have used the event to initiate a norm that making connections between mathematical concepts is an important part of understanding mathematics (Kazemi & Stipek).

During whole-class discussions of solutions, teachers may ask students to contribute solutions different from those already given. A sociomathematical norm of what constitutes a different solution depended on how teachers and other students responded when a solution was offered (McClain & Cobb, 2001). Low-press teachers observed by Kazemi and Stipek (2001) accepted solutions that differed only superficially, but not mathematically, while high-press teachers directed students to explain how solutions differed. Similarly, the first-grade teacher observed by McClain and Cobb accepted all explanations when she requested different solutions early in the school year. When she decided to hold students accountable for contributing solutions that were different, her responses to their contributions and her notations to help her keep track of their contributions helped them distinguish solutions that were different. Afterwards, the sociomathematical norms of what counts as efficient or elegant solutions became easier to establish because students were listening for differences between solutions (McClain & Cobb).

McClain and Cobb (2001) observed the development of a sociomathematical norm concerning what counts as a sophisticated solution. Sophisticated solutions are solutions that use a more conceptually advanced mathematical idea (Yackel & Cobb, 1996). In this class, the teacher initiated the norm by her response, which implicitly demonstrated greater value for one student's solution than previous solutions. Other students contributed to the establishment of the norm by contributing solutions that could also be considered sophisticated. Lampert (1990) observed student use of efficient or elegant solutions without understanding them. In her fifth-grade classroom, students used procedures offered by their peers to solve problems involving fractions because the procedures appeared to efficiently produce the correct answers. However, Lampert insisted they use only the procedures they could connect to the representations they created.

Sociomathematical norms also regulate arguments in discussion-based classrooms. High press teachers asked students their reasons for agreeing or disagreeing with others (Kazemi & Stipek, 2001), while in collaborative groups students were expected to reach a consensus through mathematical argumentation by listening and responding to their peers' mathematical explanations (Yackel & Cobb, 1996). Cassel and Reynolds (2002) contrasted the arguments of two students in a second-grade classroom. One student offered her solutions, but would not listen to her peers' responses to her explanation. The other student listened carefully to solutions and used the language of the other person as he argued. Wood (1999) suggested that student listening plays an important role in argumentation and that teachers can help students develop effective listening habits.

Another sociomathematical norm relates to classroom responses to student errors. Responses that exploit the learning opportunities in errors provide students with a chance to change their understanding of the problem or to change strategies (Borasi, 1994; Kazemi & Stipek, 2001). Teachers in high press classes used student errors as an opportunity to have all students reinvestigate the problem (Kazemi & Stipek). Borasi suggested errors should be used as "springboards for inquiry" and described the use of a high school student's solution, which the student believed to be wrong, to help the rest of the class understand why a unique circle could not be found using only two points. In addition to helping students refine their understanding of concepts, Borasi suggested that when students explored their errors, they gained a better understanding of the nature of mathematics. How students know they are in error may be guided by sociomathematical norms regarding mathematical representations. When investigating a problem about the changing areas of rectangles, a middle school student observed by Hershkowitz and Schwarz (1999), reconsidered her reasoning when the graphical representation given by a computer contradicted her hypothesis.

Cobb et al., (1997) described classroom discourse in which students' mathematical activity subsequently became the object of discussion. In an activity of partitioning numbers, both the teacher's question asking students how they knew they had all possibilities and the tabular record she used to keep track of their ways supported students' reflection on their previous activity. Cobb et al. referred to this type of discourse as reflective discourse and suggested that it supported student's mathematical disposition.

Similarly, a middle school teacher observed by Hershkowitz and Schwarz (1999) asked students to explain the process they used to develop a hypothesis concerning the maximum volume of a box. She then focused class discussion on the process of hypothesizing to help students understand that the quality of a hypothesis did not depend on whether it gave a correct solution, but on the reasons used to form it. The authors concluded that the discussion resulted in new classroom norms and affected student beliefs about mathematical objects.

The sociomathematical norms presented above are thought to affect student learning: when students are expected to have a different solution than those already presented, they must listen to others' explanations and compare them with their own solutions, providing an opportunity to reflect on their own solution and others', a higher-level cognitive activity (Wood, 1999; Yackel & Cobb, 1996). Discussion and arguments about mathematical concepts and practices may help students refine their understanding (Borasi, 1994), and errors give students a chance to reflect on their assumptions or rethink their solutions (Cassel & Reynolds, 2002). In addition, when students witness more conceptually advanced solutions than they have given, the sociomathematical norm of what constitutes a sophisticated or “easy” solution may lead them to try more advanced ways of solving problems (Cobb, et al., 1997; McClain & Cobb, 2001; Yackel & Cobb).

Sociomathematical norms related to justification of a solution require students to relate their justification to mathematical concepts and representations and defend their strategy (McClain & Cobb, 2001; Yackel & Cobb, 1996). As students prepare to justify and defend their thinking, they are often compelled to think deeper about their solution. This gives them an opportunity to consider their explanation as an object of reflection since they need to think about how others might make sense of it (Cobb et al., 1997).

The initiation and maintenance of sociomathematical norms changes the culture of the classroom, the types of communication supported, and the beliefs of students about what mathematics is, how it is learned, and their roles in learning (Yackel & Rasmussen, 2002). Table 2 summarizes the sociomathematical norms discussed. Sociomathematical norms affect the nature of communication (Brendefur & Frykholm, 2000); the next section considers communication in more detail.

Communication

Wertsch and Toma (1995) discussed each utterance in a classroom as having both a univocal and dialogic function. The univocal function has to do with conveying meaning while the dialogic function is concerned with generating new meanings. The authors emphasized, though, that certain patterns of communication in the classroom are either grounded in the univocal function or in the dialogic function. For example, in the pervasive Initiation-Response-Evaluation (IRE) pattern documented by Mehan (1979) the primary purpose of a student’s response is to convey information to the teacher, a univocal function (Wertsch & Toma). In contrast, when students engage in mathematical argumentation with each other and the teacher, they use each other's statements as objects of reflection that can generate new meanings, a dialogic function.

Whether communication is principally univocal or principally dialogic depends in part on the goals and beliefs of the teacher (Cazden, 1988/2001, Wertsch & Toma, 1995). Van Zee and Minstrell (1997) studied communication patterns in Minstrell's physics class. Minstrell's goal was to engage all students in reflecting on the ideas of their peers. In order to encourage students to think about their peers' ideas, he responded to students' answers in a neutral manner, and also shifted responsibility for thinking back to the students, a move the authors called a *reflective toss*. Minstrell strived to follow student thinking whenever it was appropriate. Similarly, Herbel-Eisenmann and Breyfogle (1997) suggested interaction patterns should support the teacher's goal of helping students make connections and use multiple representations. The dialogic function of communication was supported when a teacher used questioning to probe student thinking, ask students what they meant, and ask students if they agreed with others (Herbel-Eisenmann & Breyfogle; van Zee & Minstrell).

Cazden (1988/2001) discussed shifting classroom communication away from teacher authority and toward more student-involved discussions focused on student ideas. This type of communication requires longer teacher wait-times (Cazden). Cazden also described scaffolding to provide a temporary support, but emphasized that students do not internalize knowledge exactly as the teacher suggested but may discover new processes: "there is a critical difference between helping a child somehow get a particular answer and helping a child gain some conceptual understanding from which answers to similar questions can be constructed at a future time" (Cazden, 1988, p. 108).

This difference was also the focus of Wood's (1998) discussion of funneling versus focusing. Funneling occurs when a teacher takes on the cognitive aspects of a problem and asks a series of simple questions to guide a student to a correct answer. In this type of interaction, the student uses cues to determine the answer the teacher is looking for without engaging with the mathematical concepts of the problem. In contrast, focusing refers to the use of teacher questions to focus class attention on specific mathematical ideas and solutions provided by students (Wood). However, even when a teacher's question is aimed at conceptual understanding, students may still not think about the concepts. Vinner (1997) described this type of behavior as pseudo-conceptual: "In mental processes that produce conceptual behaviors, words are associated with ideas, whereas in mental processes that produce pseudo-conceptual behaviors, words are associated with words; ideas are not involved" (p. 101).

Sociomathematical norms support communication that focuses student attention on mathematical concepts and provides occasion for them to continue thinking (Cobb et al., 1997). Brendefur and Frykholm (2000) distinguished types of communication as either supported by social norms or supported by sociomathematical norms. They suggested that social norms support the teacher as mathematical authority and transmitter of understanding and contrast this idea with sociomathematical norms that support students' agency as the source of mathematical ways of knowing. Table 3 illustrates the types of communication included in their framework.

At the least interactive level, Brendefur and Frykholm (2000) described uni-directional communication as characterized by teacher lectures. This type of communication functioned univocally and included closed questions posed by teachers, but student answers did not provide information about student thinking and was not used by teachers to develop concepts. At the next level, also primarily univocal in function, contributive communication defined discussion that emerged when the teacher or students assisted other students, or when students presented their solutions, but classroom social norms did not oblige students to listen to each other. Reflective communication referred to discussions in which the teacher and students reflected on ideas offered by students who used these as opportunities for deeper explorations. This happened as students attempted to justify or refute conjectures, and was supported by sociomathematical norms in which students were expected to listen to each other and respond thoughtfully. Finally, instructive communication occurred when teachers provided situations that stimulated students to reconsider their current understanding of the concepts, possibly modifying it. In addition, the resulting communication informed the teacher of students' current conceptions and was used to further develop instruction. Reflective and instructive communication emphasize the dialogic function of communication. Yackel and Cobb's (1996) experimental classroom provided many opportunities for instructive communication since the teacher posed situations that became problems for students because they were not given algorithms to find the answers.

Students may also let teachers know what they are thinking by asking a question. A study of college students' questioning of their instructors indicated that how receptive instructors were to students' questions influenced the likelihood students would frame questions (Karabenick & Sharma, 1994). Student questioning showed students were engaged, but since so few students ask questions in college classes, the authors posited that students believed teachers did not want them to ask questions, and that students were afraid teachers would respond to their question with a question they could not answer. In a related idea, Walen (1994) found that high school students who did not receive direct answers from a teacher because the teacher wanted them to think found the approach frustrating, and indicated they believed that the teacher's job was to provide clear answers.

Besides questioning to engage students in thinking, teachers may provide rich mathematical tasks during class. Although teachers presented tasks that encouraged students to reflect on and modify their current understandings, students' implementation of the tasks in some cases did not maintain the level of cognitive demand intended by the task writers (Henningsen & Stein, 1997). For example, when students worked on tasks using memorized procedures without connection to concepts, the cognitive demand was low. Alternatively, when they work on a task by applying procedures meaningfully to concepts they understand, or by making and testing conjectures or framing problems, the cognitive demand remains high (Henningsen & Stein).

In their examination of tasks used in middle school classrooms, Stein, Grover, and Henningsen (1996) documented classroom factors contributing to a decline in cognitive demand as students worked on the tasks. A major factor was the inappropriateness of the task for students because students lacked the necessary prerequisite understanding. Other times, when students seemed to be struggling, teachers would take over the more difficult parts of the task so students could complete it successfully. The study also found factors present when students maintained a high level of cognitive demand, including the appropriateness of the task in relation to students' prior knowledge, appropriate amounts of allotted time, high-level performance modeled by the teacher or students, and sustained pressure for explanation and meaning through teacher questioning, comments, and feedback.

However, (Lobato, Clarke, & Ellis, 2005) found that under certain circumstances, teacher telling could provoke student sense-making. Although telling has been perceived as characteristic to traditional teaching and indicative of teaching and learning as transmission of knowledge, Lobato et al. argued that the function, rather than the form, of a teacher's utterance is what provokes student sense-making. They provided examples of questions that did not stimulate productive thinking, such as those used in the funneling pattern, and examples of telling that generated sense making. Lobato et al. provided an example of a student who was trying to determine the rate a faucet was leaking given that it dripped 16 ounces of water in 24 minutes. After the student gave an incorrect answer with incorrect reasons, the teacher asked several questions to determine the student's current understanding of division. The teacher then initiated by drawing a diagram with both units and explained the connection between partitioning and dividing; she did not provide the student with a procedure. Eventually the student was able to solve similar rate problems and explain her reasoning.

Lobato et al. (2005) concluded the function of telling was determined by the teacher's intentions when making the statements, the nature of the telling, and students' interpretations of the statement. Furthermore, the researchers examined telling by considering whether the content was procedural or conceptual, and by considering the context of the statement with respect to recent interaction. They demonstrated that telling could promote students' conceptual development and concluded that telling could fit into a constructivist view of learning by considering how students interpret the statements.

Summary

Cobb and Yackel (1995) provided a connection between communities, interactions related to mathematics, and student learning. Specifically, the authors determined that social norms related to mathematical interaction could affect students' beliefs about their roles and classroom social norms, and also provide learning opportunities as students reflected on their mathematical ideas and the mathematical ideas offered by their classmates. Brendefur and Frykholm (2000) extended the literature on sociomathematical norms by connecting them to particular types of communication, positing that social norms may contribute to lower-level communication, while sociomathematical norms that focus on students' explanations of mathematical concepts support higher-level communication. However, they did not research this idea (Brendefur & Frykholm). The present study endeavored to understand how social norms and sociomathematical norms affected types of communication.

While there is extensive literature on sociomathematical norms and the opportunities for learning they provide, most of the research on sociomathematical norms has been conducted in elementary and middle school classes. The research on sociomathematical norms were extended to college classes by Rasmussen, Yackel, and King (2003) and Yackel, Rasmussen, & King (2000) when they observed college differential equations classes. The authors provided careful descriptions of students participating in the sociomathematical norms; however, the researchers did not say if all the students in the class participated at the level described. The current study considered the participation of all students in the two classes who participated in the study.

The literature on sociomathematical norms also provide a connection between community and students' ways of knowing since sociomathematical norms affect students' beliefs about the nature of mathematics and their roles in learning with understanding (Cobb & Yackel, 1995). Students' ways of knowing stem from their beliefs about the nature of mathematical knowledge, so affecting those beliefs change their ways of knowing (Baxter Magolda, 1992).

Baxter Magolda (1992) considered both Perry's (1970) framework of male college student intellectual and moral development, and Belenky et al.'s (1986) framework of women's ways of knowing when they created a framework of college students' ways of knowing. Boaler and Greeno (2000) also used Belenky et al.'s framework to study successful high school mathematics students of both genders. Their study described students' ways of knowing as resulting from the constraints and affordances of their classroom communities. In their results, Boaler and Greeno described all classes as either discussion-based or didactic as perceived and reported by the students. Since classroom communities are complex places (Chazan, 2000), where much can be lost when described in dichotomous terms, a need existed to observe student interaction within classroom communities to discern how specific factors of community affected the students' ways of knowing. In addition, Boaler and Greeno did not investigate students' ways of knowing before their current mathematics classes. I used Baxter Magolda's framework in the current study since she focused on college students of both genders and incorporated many of the ideas from Perry and Belenky et al.

The emergent perspective was the guiding perspective of my research design, data collection, and analysis. This perspective combines symbolic interactionism, ethnomethodology, and social constructivism (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1995). Symbolic interactionism and ethnomethodology were used to examine the development and negotiation of roles, social norms, and sociomathematical norms (Cobb & Yackel; Voigt, 1996) while social constructivism was used to interpret individuals' knowledge as they constructed it in social interaction and through mathematical activities (Ernest, 1996). Ethnomethodology suggested the use of student surveys to interpret the conceptions students held of themselves as mathematics learners, their ways of knowing mathematics, and the use of interviews to understand how participants' viewed their roles. In addition, ethnomethodology suggested case studies and observations to determine patterns of interaction (Krummheuer, 1995; Voigt). Symbolic interactionism suggested observations were necessary to analyze the negotiation of social and sociomathematical norms. A perspective of social constructivism also provides a reason to observe interactions related to mathematics and to collect artifacts to examine students' current conceptions of mathematical concepts.

Chapter Three: Methods

The intent of this study was to describe and analyze the negotiation and maintenance of community and ways of interacting in two small community college precalculus classes. A further goal was to examine the relationship between characteristics of the community and interactions and students' ways of knowing. The following questions guided the collection and analysis of data.

Research Questions

1. In the classes under study, what is the nature of the classroom community and how does it develop?
2. In the classes under study, what is the nature of teachers' and students' interactions related to mathematical activity?

Research Design

Participants

The cases focused on two community college precalculus classes in a mid-sized city in the northwest United States during the eight-week summer quarter of 2005. The student body at Central Community College (all place and person names are pseudonyms to protect the identities of the participants) had around 15,000 students, consisting of 60.9% female, 39.1% male, 76.0% White, 2.9% African American, 5.4% Hispanic, and 3.2% Native Americans (Washington State Board for Community and Technical Colleges [WSBCTC], 2005). The student body at City Community College had around 10,000 students and consisted of 60.9% female, 39.1% male, 80.6% White, 2.8% African American, 4.1% Hispanic, and 2.7% Native American (WSBCTC, 2005).

My primary goals in selecting the case study classes were to find introductory college-level classes taught by instructors who had an excellent understanding of the mathematics, and who valued community and encouraged interactions in their classes. I chose Precalculus I and II as the introductory classes since they were the first college-level mathematics classes of students who major in mathematics, science, or engineering fields, and afforded the greatest variety of student identities with regard to mathematics. I wanted introductory college-level classes because many developmental mathematics students have already decided they will not pursue higher mathematics and lack ability, while many students in higher mathematics see themselves as mathematically able. Since some science majors require only precalculus, some participants were also in their last mathematics class.

The instructors, Mr. Reilly and Mr. Anderson, were selected because of their efforts to establish community and relationships with their students, level of classroom interaction, and their reputations for strong mathematical knowledge and standards. The teachers agreed to participate, understood the goals of the research project and signed consent forms indicating their understanding that participation was voluntary (Appendix A). Both instructors were involved in local and state efforts to create standards for students entering college-level mathematics prior to and during the study, and I attended meetings with them in this capacity.

Mr. Reilly and his class. Mr. Reilly's education and experience consisted of a Bachelor of Arts in Mathematics Education, an M. S. in Mathematics, and eighteen years of teaching experience, most at Central Community College where he earned tenure, and a few years at a local high school. He had been a student at Central Community College and described a recently retired teacher there as instrumental in his choice to major in mathematics. Mr. Reilly's students consistently evaluated his teaching as very effective. The entrance to his office and five other offices were along the walls of a large study room containing many study tables and whiteboards. His office was in the center of one wall and students working at the tables easily approached him when his door was open. He taught two other classes daily during the term of the study, starting at 7:30, with no breaks between classes until after this class. He had office hours immediately after this class, at 12:30.

Students and colleagues of the researcher who had taken classes from Mr. Reilly reported that his classroom was highly interactive, with most of the interaction teacher-to-student. One student reported that he was inspired to major in mathematics after taking a class taught by Mr. Reilly and described his reasons based on the relationship Mr. Reilly established with his students. When describing the teachers who made the biggest impact on his own teaching philosophy, Mr. Reilly related a story about a teacher who remembered his name two years after having him in class. When I observed Mr. Reilly teach prior to the present study, he seemed relaxed, made eye contact frequently with his students, joked with them, and appeared to really like his students.

Using the textbook, *Precalculus: Functions and Graphs, 9th ed.* (Swokowski and Cole, 2002) the course covered most of the sections in chapters five through seven, nine, and ten: Trigonometric functions, identities and formulas, applications of trigonometry, sequences and series, mathematical induction, the binomial theorem, conics, polar coordinates and parametric equations. Notes in the front of the text highlighted the features of the text: illustrations provide brief demonstrations of the use of definitions, charts give students easy access to summaries of properties, examples provide detailed solutions of problems similar to those that appear in exercise sets, step-by-step explanations, discussion exercises, graphing calculator inserts, exercises, and real-life examples, "boxed guidelines enumerate the steps in a procedure or technique, to help students solve problems in a systematic fashion" (Swokowski & Cole, , p. *xi*).

The summer quarter consisted of thirty class days, each an hour and a half; three days were used for taking exams, while one exam was taken in the Mathematics Lab. Students were assigned homework which was collected on exam days and awarded ten points as long as it was at least two pages long and had their names on it.

Eighteen students originally enrolled in the course, and fourteen agreed to be participants in the study, five female and nine male (see Table 4 for student demographics). One participant, Julie, dropped the course after the first two weeks. Participants ranged in age from 19 to 42 years old with a mean age of 26 and a median age of 25.5.

Mr. Anderson and his class. Mr. Anderson held tenure at City Community College. He had a Bachelor of Arts in Mathematics Education, an M. S. in Mathematics and six years of teaching experience. His student evaluations had always been excellent and he had a reputation among his former teachers as an excellent mathematician. Mr. Anderson and I were in graduate school together working on our master's degrees and as graduate instructors for two years, and later worked together for a year as instructors at the same university before he moved to his present position as a full-time faculty member. His office was in a corner of a small grouping of offices; there were two desks with one chair each between the offices, and benches along the wall a few feet away. Mr. Anderson was very relaxed and friendly, and easy to engage in conversation about mathematics or approaches to teaching. He did not have office hours in the summer, but taught a lab class from 11:00 to 12:30 daily and invited students from this class to come see him for help at that time.

Mr. Anderson was well liked by students and colleagues. While he interacted with students during class, he also fostered student-to-student interaction. For example, as colleagues, Mr. Anderson and I worked together to create group projects for our Mathematical Reasoning classes. Mr. Anderson also valued relationships with students; one quarter when he had the office across the hall from me, he required all his students to come by his office early in the quarter to meet him and tell him something about themselves.

Mr. Anderson's class used the textbook *Precalculus: Mathematics for Calculus, 4th ed.* (Stewart, Redlin, and Watson, 2002). While the course taught by Mr. Reilly focused primarily on trigonometric functions, Mr. Anderson focused on algebraic functions and their general characteristics. The course covered most of the first four chapters and parts of chapters eight and ten: review of solving equations, modeling with equations, inequalities, graphing, lines, functions, polynomial and rational functions, exponential and logarithmic functions, one section on matrices, introduction to sequences and series, mathematical induction, and the binomial theorem. During the study, both Mr. Anderson and Mr. Reilly taught mathematical induction, and discussed concepts common to trigonometric and algebraic functions such as inverse functions and transformations of functions.

The summer quarter consisted of thirty-one class days, each an hour and a half; five days were used for taking exams. Students were assigned homework, always odd problems, but it was not collected or graded. There were three twenty-minute quizzes throughout the quarter.

Fifteen students were enrolled in the class; students' ages ranged from 17 to 37. One withdrew after the first exam, and the 17 yr-old was not a participant in the study, leaving 13 participants: 5 women and 8 men, 12 white and 1 black, including a Russian immigrant and one

Pacific Islander (see Table 5 for student demographics). Participants ranged in age from 19 to 37 and with a mean age of 24.6 and a median age of 22.

The researcher. My researcher role was best described as an observer as participant. Adler and Adler (1994) describe the observer as participant role as “those who enter settings for the purpose of data gathering, yet who interact only casually and nondirectively with subjects while engaged in their observational pursuits” (p. 380). Since I had a video camera set up and they signed consent forms, participants knew I was there as a researcher. I did not participate in class discussions, but had informal discussions in addition to semi-structured interviews with participants outside of class.

My education and background consisted of a B.S. in Mathematics, M.S. in Mathematics, a secondary teaching certificate, three years experience teaching K-12 mathematics and computers, eight years experience as a college mathematics instructor teaching algebra, precalculus, calculus, mathematical reasoning, and content courses for preservice elementary teachers. I also supervised and coordinate graduated instructors and adjuncts teaching intermediate algebra, provided student support services to disadvantaged students through a TRIO grant, and worked with faculty members to provide professional development to K-12 teachers. Prior to the present study I had worked on a Ph.D. in Education for three years and tried to incorporate many of the ideas from the literature into the classes I taught, such as using questioning strategies and having students communicate their reasoning orally and in writing.

As a student, my undergraduate and graduate mathematics classes were almost exclusively traditional-lecture format. Some teachers asked questions during the lecture, but the same couple of students provided all the answers; I was usually one of them. The questions could usually be answered with a short answer and did not promote further discussion. My teachers often explained and developed concepts, and concepts were developed in textbooks, so I did not attempt to memorize procedures without understanding concepts or making connections. Thus, I did not believe that the description 'traditional' necessarily means all students were memorizing procedures. However, I wonder if I would have chosen mathematics as a major if my earliest college teachers taught mathematics as procedures without understanding. In addition, I do not believe the nature of the mathematics I learned was as deep or connected as it could have been in more discussion-oriented environments.

I do not remember a single instance of collaborative learning inside of class and very little interaction between students; I generally avoided working with others outside of class because of time management and because I did not believe it would be useful to me. I thought many of my teachers were very good teachers, but I did not believe my success or failure depended on their teaching. I adapted my learning strategies based on what the teacher had to offer. For example, my Calculus II teacher was a graduate instructor whose English was very poor and who talked to the board, so I read the book and did more problems than were assigned in each section. I knew how well I understood the mathematics and worked to understand it completely; "completely" meant I could do almost all of the problems in each chapter and knew what I was doing and why I was doing it. This knowledge was sufficient to maintain a high GPA and graduate with honors.

My earliest teaching consisted of lectures in which I presented carefully sequenced examples, asked questions to make sure students were with me or to anticipate next steps, and encouraged students to ask questions. When students asked questions, I answered by explaining why I was doing what I was doing and how they could know what to do based on the problem situation, often making connections and distinctions between concepts. However, I did not ask *students* to explain or to make connections, since I assumed they understood when I explained it to them. Over the years, I changed my teaching practices to provide more opportunities for collaboration by presenting problems for students to discuss in small groups, expanding the types of questions I asked, promoting discussions, and extending my wait time. Over the years I became more conscious of the importance of classroom community and strive to entice all students to actively participate in discussions. My goal has been to use students' thinking to develop concepts, but I have not always achieved the character of teaching I sought.

Participant confidentiality. To maintain confidentiality, participants were assigned codes; the master list matching names was kept separate from the data. Participation was requested from all students on the first day of class through consent forms that sought permission to observe, collect copies of student work such as graded exams and quizzes, survey, and interview (Appendix B).

Data Collection

The main source of data collection was classroom observations. However, student questionnaires and teacher interviews done in the beginning helped me understand participants' goals and initial attitudes towards interaction in class. Teacher and student interviews throughout the study provided additional data regarding participant perspectives on interactions and how they related to students' efforts to make sense of the mathematics they were learning. Appendix C provides a timeline of the research process indicating when data was collected.

Student Questionnaires. Student Questionnaires (Appendix D) were given to students on the first day of class. The questionnaires were field-tested in a precalculus class, adjusted, field tested in a different precalculus class and adjusted prior to the study. Items were altered when students' responses did not net helpful information. For example, two items on the original questionnaire targeted students' beliefs about the roles of the teacher and students, but students only responded generally that their role was to learn and the teacher's role was to teach. So, the word "role" was removed and items were rephrased to more specifically address what students and teachers could do to help students learn.

The purpose of the questionnaire was to understand perceptions individual students had at the beginning of the quarter concerning how mathematics was learned and what they believed to be appropriate roles and interactions for the members of the mathematics classroom community. Other questions focused on students' identities as mathematics learners. Information from the questionnaires was used to choose students to interview so that students with a variety of perspectives were included (Tobin, 2000). However, because of the short term and students' availability I was only able to conduct three taped interviews although I had informal conversations with several other students.

Instructor interviews. Open-ended instructor interviews were conducted near the beginning of the quarter to understand instructors' philosophies of teaching and learning mathematics, the roles they believed instructors and students should have in the classroom community, and what they believed about the nature of teacher-student and student-student interactions (Appendix E). Interview questions were field-tested on an instructor with similar teaching and educational background and questions added. Interviews were audio-taped, transcribed, and coded by the researcher (Tobin, 2000). Additional open-ended interviews took place later in the quarter to check my interpretations of teachers' meanings.

Artifacts. Artifacts such as copies or digital photographs of student notes, teacher handouts, homework, exams, and quizzes were collected and stored with the research data. They were used to look for evidence of students' sense-making or applying procedures without understanding. Artifacts were also used to understand the types of tasks assigned by the teachers and the mathematical practices they targeted, the students' interpretations of what was expected of them, and how the teachers responded. Analytic memos were written for each artifact (Tobin, 2000).

Student interviews. Aligned with interpretivist perspectives, students were chosen for interviews on a dialectical principle. "To provide experiences that will advance understandings of the issues…. One criterion that often is employed is to choose someone who is most different from the first participant selected and studied…a dialectical principle is applied in the selection of participants so that the diversity of a given community is reflected in the data sources scrutinized during a study." (Tobin, 2000, p. 489). Thus, I chose the most interactive students and the least interactive students to interview, students who attended every day and students who attended less often, students who enjoyed friendly relationships with the teacher or other students, and students who rarely interacted with the teacher or other students. Specific questions depended on why the particular student was chosen for an interview. Appendix F contains the Student Interview Protocol containing questions focusing on student perceptions of interactions and ways of knowing.

The open-ended interviews were used to understand students' perspectives of the nature of community, interactions, and the extent to which they were striving to make sense mathematically. Interviews were used to clarify my interpretations of students' interpretations of the interactions, their dispositions toward learning mathematics, and to provide feedback on and promote reflection about what I observed. All interviews were taped, transcribed, and coded; analytic memos were written to provide thick description about student disposition and content of the interview.

Classroom observations. Observations were used to address both research questions. I observed each class at least five times in the first two weeks of the term and at least twice a week throughout the rest of the term, for nineteen observations per class. Observations were videotaped and a classroom observation form (Appendix G) completed for each observation. Since a goal of the study was to understand the development of community, understanding teacher and student negotiations of social norms, authority, and roles was important. Thus, the first five videotaped observations in each class were transcribed. Thereafter, I videotaped the class and reviewed the tapes after each lesson while typing and adding to my field notes. I completed the classroom observation form as I reviewed each tape and used it to determine which parts of the videotape related to the identified themes and transcribed, coded, and analyzed those parts. Field notes were taken during each observation, with details added and analytic memos written while memories of the observation are still fresh (Tobin, 2000). Limitations on observations restricted most observational data to public discourse, communication intended for all members of the classroom to hear, so I did not hear most of the conversations between students.

The classroom observation form had four parts. The first part focused on the nature of the classroom community and contained data that evidenced the roles students and teachers negotiated, who or what had mathematical authority, students' identities within the classroom, the significance afforded certain activities, and the social norms and relationships that evolved (Boaler & Greeno, 2000). The second part focused on the mathematical activities and the learning of mathematical practices and content (Henningsen & Stein, 1997; Yackel & Cobb, 1996). The third part focused on the negotiation of sociomathematical norms as found in the literature (Cobb et al., 1997; Hershkowitz & Schwartz, 1999; Kazemi & Stipek, 2001; Yackel &

Cobb, 1996; Yackel & Rasmussen, 2002) while the fourth part concentrated on types of communication (Brendefur and Frykholm, 2000; Goos et al., 2002). Table 6 summarizes the data collection techniques and their relationship to the research questions.

Data Analysis

Data analysis began when the first data was collected and continued systematically throughout the quarter; this preliminary analysis helped me determine which students to interview, the questions that needed to be asked in interviews of teachers and students, and what interactions to examine (Tobin, 2000). An initial goal of analysis was to identify key topics and investigate their characteristics and the contexts in which they occurred, then to understand these topics from the perspectives of the participants (Woods, 1992). After each observation or interview I looked for ways the data fit into categories related to the research questions, modified categories when necessary, and created new categories and codes as they emerged (Strauss & Corbin, 1990). When new conceptual categories arose, previous data was reexamined to look for evidence and questions added to the interview protocols. I used data matrices and concept maps to understand the data within the categories (Miles & Huberman, 1994). Qualitative software was used to organize and analyze data in text form (*N6*, QSR, 2002).

Analysis was also guided by sub-questions of the two research questions.

Sub-questions of Research Question 1 included:

a. What expectations concerning community did teachers and students have as the quarter started?
b. What was the nature of social norms and how did they develop?
c. What roles did instructors and students play in the community?

Sub-questions of Research Question 2:

a. What was the nature of sociomathematical norms and how were they negotiated?

b. What levels of interaction did instructors and students initiate and how were the levels maintained or reduced?

c. What was the nature of mathematical tasks and what practices did students employ while working on the tasks?

d. What did participants' interactions indicate about their beliefs regarding mathematics and learning mathematics?

To aid analysis, I maintained a journal and wrote analytic memos (Woods, 1992). Memos were used to illuminate and distinguish concepts and to note ideas for research strategies such as interview questions (Maxwell, 1996). Each memo was dated, coded and stored with the data.

Since interpretive research seeks to understand the diversity of a community, early and systematic analysis of data allowed discrepant data to be sought and alternative explanations to be formulated (Tobin, 2000). I looked for evidence in observations that appeared to refute my initial conceptions.

An important goal of this study was to examine the relationships between components of community, classroom interactions, and student learning using an emergent perspective. To accomplish this, both within-case and between-case data matrices to compare and contrast themes were employed (Miles & Huberman, 1994). Between-case matrices were used to contrast factors between the two classrooms and within-case matrices were used to examine connections between factors related to community, factors related to interaction, and students' ways of knowing. In addition, it was important to understand the nature of the context of each interaction since the situation and perspectives within the situation relate reflexively (Woods, 1992). I analyzed and present data in ways that provided interpretations of participant interactions and interpretations. This was accomplished by attempting to view the interactions as the participants did and by providing thick descriptions that may give insight regarding participants' purposes (Schwandt, 1994).

Figure 1 shows the categories used to code the data and Table 7 gives the descriptions of each category as used in N6 (2002). The categories and their descriptions were obtained from the literature and themes that arose during data collection. While most categories were used just as they were described in the literature, Boaler and Greeno (2000) described two types of mathematics classroom communities important to this study, "didactic" and "discussion-based," each of which included several characteristics. "Didactic" included students watching teachers present procedures, then practicing alone, students were not expected or encouraged to discuss ideas, and the goal was for learners to memorize procedures (Boaler & Greeno). These characteristics were captured in other categories such as teacher role, mathematical authority, student role, listen, procedures, memorizing, student-student relationship, procedural, and instrumental. Likewise, "discussion-based" communities included student roles of discussing questions as a class and contributing to shared understanding; students were able to work with classmates and described relationships as central to their learning (Boaler & Greeno). Categories describing discussion-based communities included student role, active, discussion, student-student relationship, student explanation, and relational.

Quality and Ethics of Research

One of the instructors was my friend, while the other was an acquaintance; I worked with both instructors on state-wide initiatives to improve high school students' preparation for college mathematics. I feel indebted to both instructors for allowing me to study their classes and have imagined how I would feel if the teaching and learning within my classes were explored and

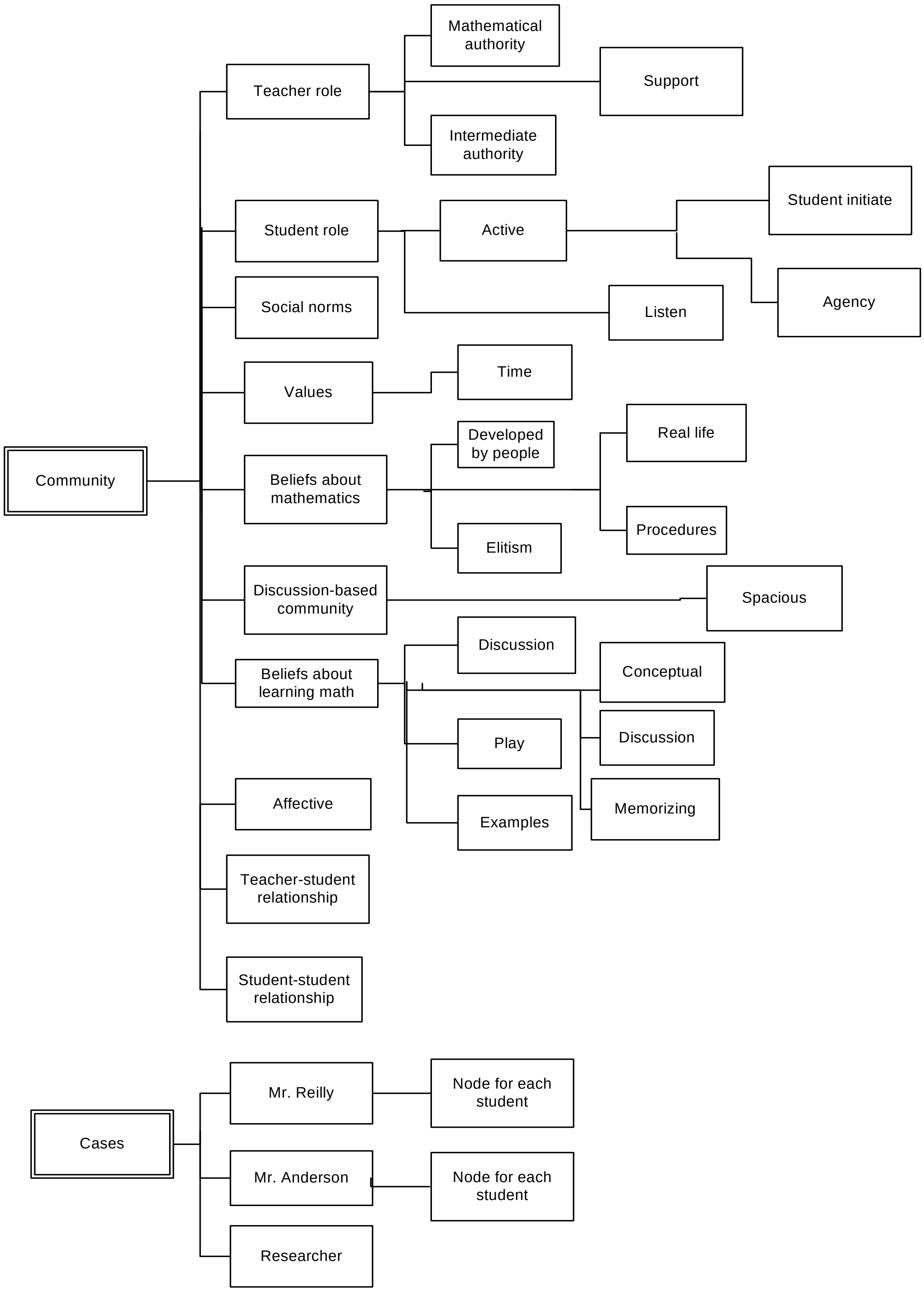

Figure 1: Hierarchy of Categories

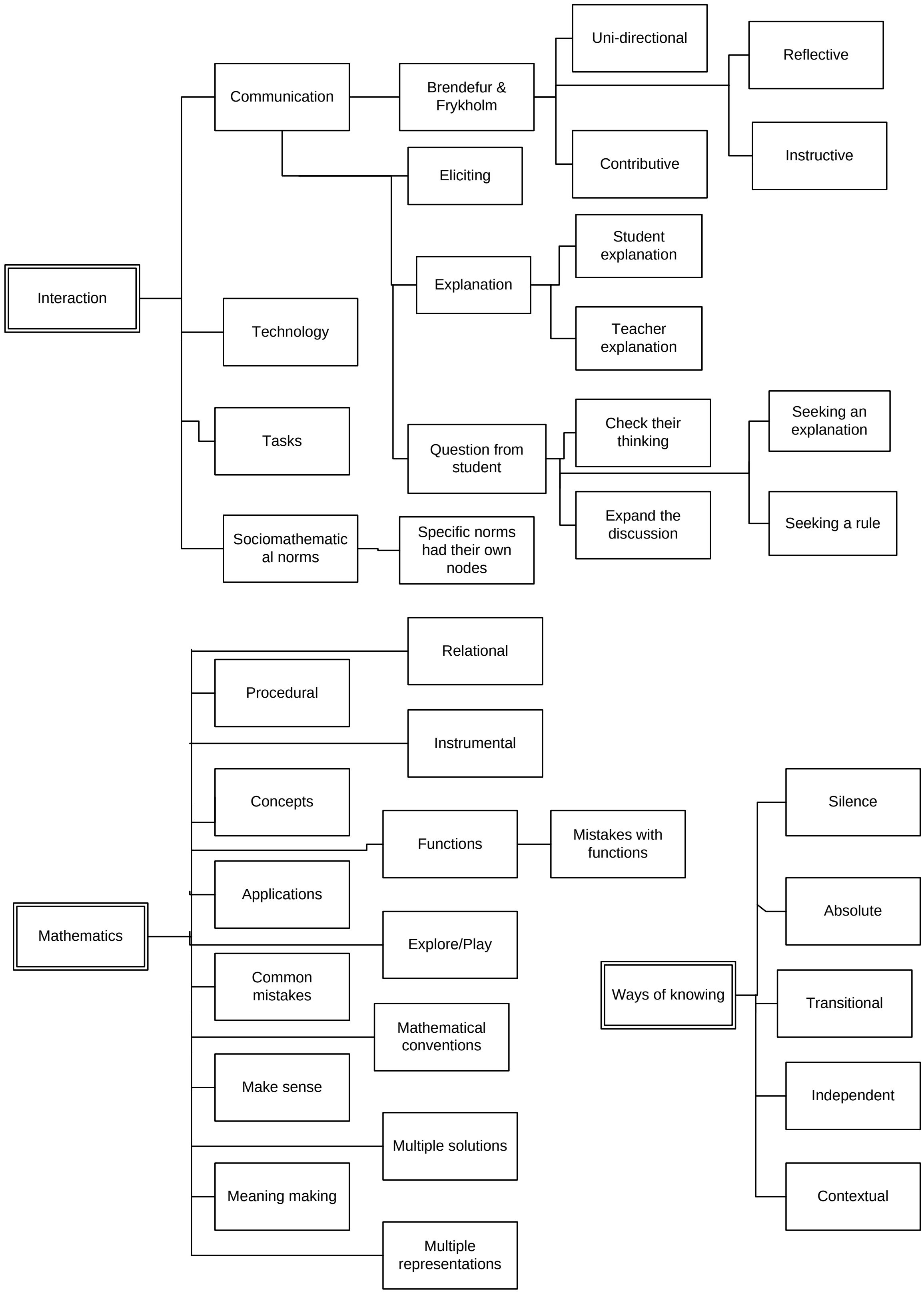

Figure 1 (continued)

written about in detail. I did not want to alienate the teachers during the project, nor did I want to feel like I betrayed them when the project was finished. On the other hand, both teachers indicated they were interested in the findings and wanted to grow from the experience. In order to maintain an open-mind during analysis, I used analytic memos to explore how this issue affected how I collected and analyzed data, and used the insights gained to work toward fair-mindedness. In addition, I used peer debriefing as mentioned later in this section; my advisor did not know the real identities of the teachers.

Interpretivist frameworks include criteria and methods for establishing the quality of research including credibility, authenticity, transferability, and generalizability (Tobin, 2000). Credibility refers to the believability of the assertions and was established by multiple sources of data. I presented interpretations to participants to determine if their meanings were interpreted accurately, and included evidence of how existing literature was related to the study. Both researcher and participant views were included since these views provide readers with a more complete understanding of the interactions.

Data was member-checked to test the accuracy of researchers' interpretations (Tobin, 2000). Both instructors affirmed that the evidence and analyses accurately represented the classes I observed. Member checks also provide evidence of authenticity (Tobin). I maintained a researcher journal that detailed how my understanding of the research changed throughout the study (Tobin). I worked closely with my dissertation advisor, Amy Roth McDuffie, to confirm or refute interpretations from data analysis. Her research program focuses on using qualitative research methodologies in mathematics education. Validity was also established by interviewing students with varying perspectives of the nature of the community and classroom activities. I helped students become comfortable telling me their perceptions by ensuring confidentiality and

I looked for evidence to contradict or support their perceptions in my observations. As part of my position prior to this study, I regularly engaged individual students in conversations about their classroom activities and instructors and they were usually able to be critical even if they liked the teacher.

Case studies are not transferable in the sense that positivistic research has traditionally used the term to indicate broad transference to similar situations, but rather as depending on the reader to see the case as similar to their own situation (Tobin, 2000). To provide transferability I will provide sufficient description and detail of the contexts so that readers may determine whether the evidence and assertions apply to their situations (Tobin).

The ethics of this proposed research concerns the protection of the reputations of the teachers and students. I chose these teachers because they have excellent reputations within the mathematics teaching community in this city and I hope to be able to learn how the community and interactions that develop in their classrooms were related to students ways of knowing mathematics. These teachers have less to fear than would teachers with poor reputations or teachers without tenure. They were assured of the confidentiality of the data and reports generated by this study since only pseudonyms will be used for the school and participants. The students will have no reason to fear repercussions from the teachers since their identities were kept confidential.

Chapter Four: Findings in Mr. Reilly's Class

Background

Located on a quiet hall next the Math Lab, the classroom looked new and contained six neat rows of six or seven desks each. The rows were close together and pressed against one side wall and the back wall, leaving no room to walk around. The whiteboards and chalkboards looked cleaned each day.

Several students usually gathered in the hall and talked before class as they waited for the door to be unlocked. Tim was often the first student there and I talked with him many times. Natalie was also usually early and willing to talk with me informally and in a taped interview. Fourteen of the eighteen students enrolled in this class agreed to participate in the study.

Research Question 1: The Nature and Development of Community

In the following sections, I address the expectations students had of their roles as they began the course and how they enacted their roles throughout the quarter. I characterize Mr. Reilly's roles, how students and instructor negotiated and maintained social norms, and describe the nature of mathematics as it was portrayed during class.

Students' Expectations and Enactments of Their Roles

Students completed the Student Questionnaire (see Appendix D) at the beginning of the first day of class. I discuss the results of the survey, which provide evidence of the types of roles students believed would help them learn, then describe how they maintained or changed those roles in response to affordances and constraints of the classroom environment.

Students' expectations of their roles. Six questions on the questionnaire specifically focused on the roles of students in the class (Questions 6, 7, 8, 9, 10, and 13). Question 6 addressed what students do during class to help them learn. The majority of students responded, “take notes” while some students included doing examples, watching the board, and practice outside of class (see Table 8). Only two students mentioned interaction with others: “Being attentive, ask questions, do the assignment (persistence)” (Mark Questionnaire, June 27), “Take really good study notes, have a study partner” (Shawna Questionnaire, June 27). Mark included asking questions of the instructor in his role, while Shawna valued having a study partner. However, most student ideas of their roles in mathematics classes were to transcribe the instructor’s ways of doing problems and then practice those problems or methods. Their responses strongly resembled the description provided by Smith (1996) typifying the learning of mathematics in the United States: “Students learn by listening to the teachers’ demonstrations, attending carefully to their modeling actions, and practicing the steps in the procedures until they can complete them without substantial effort” (p. 391).

However, two questions on the questionnaire specifically targeted student ideas about the value of learning from and getting to know other students. When asked if they liked to work with a partner or in a group, four students responded they preferred to work alone, but of those who wanted to work with others, common reasons were to check answers or get help on their homework (see Table 9). The other question addressed whether they wanted to get to know other students in the class (see Table10), eight students responded affirmatively, “Yes, helps with studying” (Jake Questionnaire, June 27), although six students did not say why. Three students indicated they wanted to know the instructor but not other students. Their responses show the majority of students valued knowing each other but did not suggest they could learn new ideas by listening to each other or by trying to convince each other of the validity of their mathematical ideas, suggestions by the NCTM (1991) for students’ roles in discourse.

Three questions on the questionnaire related to students' roles in class discussions. The majority of students wanted the instructor to involve the class in discussions (see Table 11), and when asked what types of input they would provide, most students responded they would ask questions (see Table 12), "I only ask questions if I don't understand" (Ryan Questionnaire, June 27). Only three students said they would also make suggestions or answer questions, "Yes, I ask questions and give some answers to questions" (Mark Questionnaire, June 27). Consistent with previously stated ideas, these responses resonated with the notion that students expected the instructor to provide clear explanations, and while they liked class discussions, most expected their roles during class discussion to be limited to asking questions when they did not understand.

Another question related to students' roles during class discussion addressed listening to other students' questions and explanations (see Table 13). Four students specifically mentioned listening to other students' questions as valuable to them, "Yes, most likely I'll have the same question" (Natalie Questionnaire, June 27), while four students also mentioned they liked to hear other students' ideas or explanations "Sometimes it helps if they have a different perspective on it" (Karen Questionnaire, June 27). Two students expressed concern that other students' explanations might confuse them, "Sometimes – they confuse me as well" (Julie Questionnaire, June 27). In general, more students wanted to hear their peers engage in discussion than wanted to engage in it themselves, indicating a preference for hearing others' ideas but not sharing their own to contribute to others' learning or to test their ideas, the type of discourse recommended by the NCTM (1991) and that indicate contextual knowing (Baxter Magolda, 1992).

One student's responses in particular highlighted the contrast between an unwillingness to share their own ideas and wanting to listen to the instructor and peers. Karen's response to whether or not she liked the instructor to involve the class in discussion was, "No, I'd rather listen to the teacher," and about offering input in class discussions, "No, I like to listen and take it in" (Karen questionnaire, June 27). These responses indicate Karen was as an absolute receiver (Baxter Magolda, 1992; Brew 2001). Her responses to other questions about understanding or memorizing mathematics also evidenced absolute knowing, since she replied that memorizing steps and formulas helped make it easy to learn, and replied, "I don't need to understand where it comes from, I just do it" (Karen Questionnaire, June 27). In contrast, she wanted to hear other students' explanations, "Sometimes it helps if they have a different perspective on it," an indication of transitional knowing. Her role in class remained consistent with her stated preferences; she did not speak in class during my observations.

Another student provided an example of one who did not behave consistent with his questionnaire responses. Jeremy responded to whether he liked the instructor to involve the class in discussion, "Yes, but math is not really discussion oriented; right or wrong answers," and wrote that he did not usually speak up in class unless there was something he did not understand (Jeremy questionnaire, June 27). However, after the first few days of class Jeremy regularly volunteered ideas during problem solving discussions (e.g. Fieldnotes, July 6; Fieldnotes, July 19; Fieldnotes, July 27; Fieldnotes, August 16). His questionnaire responses indicated his perception of mathematics as a set of procedures to find correct answers was not amenable to discussion, and his role in discussion would be limited to asking questions. However, it appeared this class affected this belief and as a result, he participated in ways he did not expect.

In summary, student questionnaire responses regarding their roles of taking notes and practicing evidenced their experiences and expectations of mathematics classrooms as collections of individuals rather than mathematical communities, similar to the students in Boaler and Greeno's (2000) didactic classes and counter to recommendations of the NCTM (1991). Responses also indicated their belief that mathematical knowledge was certain, belonged to the instructor, and they could gain that knowledge by listening to him and practicing, a nature of knowledge and gaining knowledge characterized by Baxter Magolda (1992) as absolute. Although the majority wanted to get to know each other (see Table 10), the relationships were not valued for the chance to share and test ideas.

Students' enactments of their roles. Students' roles in this class consisted mostly of listening to Mr. Reilly and answering and asking questions. They also listened to each other's questions, but only once did I hear a student answer another student's question during class (Fieldnotes, July 12). Students spoke up more frequently as the term progressed, probably because when they interrupted, Mr. Reilly responded respectfully (see Appendix H, lines 31-49). In this episode, Tim first suggested that the answers contradicted his understanding, however, Mr. Reilly did not understand Tim's concern. When Tim persisted in explaining why he did not think the answers were correct, Mr. Reilly understood what he was saying, "I agree, you should be worried about that. And it's a good point…," then rephrased the problem to the whole class. Mr. Reilly used Tim's question as a "teachable moment" for the class; the format of discussion provided opportunities for student questions and comments affecting the direction of the focus on mathematics. However, the roles and social norms provided that Mr. Reilly directed discussion about the issue rather than have students debate the ideas. Tim's questioning of Mr. Reilly's solution showed evidence of higher ways of knowing since he did not just accept the solution without question. In their description of a spacious classroom, Angier and Povey (1999) also found students became confident enough to speak up when they thought their teacher was wrong.

Student participation in whole class discussion varied. One student, Susan, provided quick answers regardless of the situation, and many of her answers appeared to be guesses. She sat next to the camera, so all of her comments were audible to me while many were not audible to Mr. Reilly over other students' answers closer to him. Several times she answered in a soft voice but was the only one to answer, and he asked her to repeat her answer. The following episode provides an example of her behavior; the goal of the discussion was for students to discover the range of values output by the calculator when using the inverse trigonometric functions.

Mr. Reilly: What your calculator will do is it will take the easiest answer. So go ahead and do a point five, take the inverse sine of point five and see what the calculator says.

Susan: Thirty.

Mr. Reilly: Yep, from the calculator, okay, it will tell you the inverse sine of one-half is thirty degrees. Uh, inverse sine, from your calculator, of a negative one-half? [He writes these in a vertical list on the board; pause 6 seconds] there's a whole bunch of them too, let's see which one your calculator picks.

Susan: Negative thirty.

Mr. Reilly: Negative thirty, good, okay, sine inverse of one?

Susan: Ninety [Observer Comment (OC): there was a 3-second pause before she answered. she's answering this and the next faster than the calculator could be used].

Mr. Reilly: Ninety, good, sine inverse of negative one?

Susan: [immediately] Negative ninety.

Mr. Reilly: Negative ninety, good, do you see what your calculator's doing? [steps back away from the board] It's pretending that the second and third quadrant don't exist; that's the way it copes with it. Issue? I see no issue, what your calculator will do, anytime you hit that inverse sine button, notice that if you want to do the full range of sine, you go from bottom to top, that's all your heights… And when it said thirty that what it was really doing in a sense was giving you the reference angle. Make sense? Cosine, same thing. See what the calculator tells you about the cosine inverse of one-half.

Susan: Sixty [very quickly, no more than one second].

Mr. Reilly: Good, sixty [he repeats quickly, most of the class is still putting it in their calculators]. Okay, inverse cosine of negative one-half.

Susan: Negative sixty.

Mr. Reilly: No [he had his chalk starting to write it, but pulls away].

Susan: No? I was just guessing.

Mr. Reilly [laughs]: No, yeah, guesses is bad. It's right up there with the old assuming thing.

Susan: One twenty, one twenty.

Mr. Reilly: Okay, good, does everybody else see that? Calculator would say a hundred and twenty. If you were to ask for the inverse cosine of one? [pause 3 seconds] And basically what it's asking for is, who's a nice angle that has a cosine of one?

Student: Zero.

Mr. Reilly: Zero does, okay. If you were to do the inverse cosine of negative one?

Student: One-eighty.

Mr. Reilly: One-eighty, good. Where's the calculator looking now?

Student: Reference.

Mr. Reilly: Kind of, but,

Student: One and two.

Susan: Two and three [talking low so he probably did not hear her].

Mr. Reilly: There you go; [responding to the students at the front] good, because the *x*-axis could also be three and four. What the calculator is doing now, now cosine, unlike sine, sine is an up and down number, it's the *y*-coordinate. Cosine is on the *x*-axis, it's a left and right number. …okay. Oh, here lets see if we can reason through tan.

(Observation, June 30)

This episode lasted four and a half minutes. Although Mr. Reilly repeated several times that the answers were given by the calculators, Susan did not use a calculator and was trying to answer as quickly as possible. Her quick answers demonstrated her willingness to interfere with other students' ability to participate in the way Mr. Reilly intended. However, since it was Mr. Reilly's role to quickly respond to the first answer he heard, he inadvertently supported Susan's efforts. Susan displayed this behavior on other occasions (e.g. Fieldnotes, August 10) but no other student appeared to behave this way.

While the preceding episode was intended to be an investigation to have students discover the ranges of the inverse trigonometric functions and highlight the idea that the trigonometric functions only have inverses if their domains are restricted, in each case, Mr. Reilly told them the answers. The investigation provided more opportunity for student engagement and more encouragement for student thinking than pure lecture, but students were not afforded the time and expectation to actually make the discoveries. In addition, students' roles in this exploration also consisted of finding the answers posed by Mr. Reilly rather than designing the exploration, or even choosing the angles to help them determine the ranges of the inverse trigonometric functions. Unlike “doing mathematics” as described by Henningsen and Stein (1997), students did not have opportunities to design their own investigations during class.

The preceding episode was typical in some ways and demonstrates students’ usual roles in this class were to answer closed questions with short answers and listen to Mr. Reilly. Students sometimes answered questions using the words or ideas Mr. Reilly had recently used, and these responses sometimes appeared to be guesses. In the aforementioned episode, the class first examined calculator outputs of the inverse sine function and Mr. Reilly told them the output of thirty was the reference angle. Then, when he asked for the range of output values for the inverse cosine function, specifically after asking about the inverse cosine of negative one, Mr. Reilly asked what the calculator was doing and a student answered “reference,” even though a one-hundred and eighty degree angle would not be considered a reference angle. While the question was meant to give students an opportunity to think, the student who answered appeared to be repeating Mr. Reilly’s previous reason rather than thinking. The quick pace of lecture may have contributed to this behavior.

Several students indicated they were not striving to learn the concepts but their goal was to get through the course with a grade of a C, the minimum grade required for the course to satisfy their requirements. Susan mentioned she only wanted a C in this class (Fieldnotes, July 27). Steve reported that he did not spend much time studying for this course because his programming class required so much of his time (Steve Interview, July 21), and in response to whether he wanted the instructor to involve the class in discussion, Ryan responded, “No, I am usually tired and don't care. I am not a math major, I just want to finish the requirement with a high grade” (Ryan Questionnaire, June 27). Ryan rarely contributed in class while Susan appeared to try to hurry the pace of the class. Another student, Shawna, claimed she only wanted a C in the class (Fieldnotes, July 21), but her behavior indicated she was striving to learn. She exchanged phone numbers with Natalie early in the quarter and met with Natalie or Karen to study on several occasions (Fieldnotes, July 21). She participated in discussions and asked homework questions (Fieldnotes, August 10).

In contrast to those who just wanted to pass the class, Natalie took the class as part of the requirements to teach mathematics (Natalie Questionnaire, June 27). She spent hours working on the homework, averaging four to five hours per day studying for exams (Natalie Interview, August 10), and came prepared with homework questions, sat at the front of the class, and asked and answered many questions. These examples indicate that students' goals at the outset of the class may have influenced the level they were willing to participate.

Students' roles and ways of knowing. Students' expectations of their roles as indicated by their questionnaire responses suggested they were predominantly absolute knowers (see Table 14). Twelve of the fourteen students had more responses coded for absolute knowing than any other category. Tim had as many responses coded for absolute knowing as transitional knowing, while Shawna's responses indicated she learned from peers and strived to understand, evidence of transitional knowing, but still expected the teacher to explain the ideas. Likewise, students' roles supported absolute knowing since their main role was to listen and answer questions, and rarely included sharing their ideas (for exceptions see Observation, July 6) or responding to their peers' ideas because there were no opportunities for group work or discussion between students during class. Thus, higher ways of knowing encouraged by peer interaction, such as contextual knowing, were not supported by the roles expected by and afforded to students.

The Nature of Mr. Reilly's Roles

Questionnaire responses also indicated students' preferences in a mathematics instructor's role. In this section, I compare and contrast their preferences with the roles Mr. Reilly actually played in this class, and discuss students' responses to those roles. Finally, I examine how Mr. Reilly's roles matched and sometimes challenged students' ways of knowing.

Students' expectations of Mr. Reilly's roles. Three questions from the Student Questionnaire (Appendix D) elicited students' expectations of a mathematics instructor's role. When asked how they best learn mathematics (see Table 15), five students included the role of the instructor. Jake simply answered, "teacher" (Jake Questionnaire, June 27), while Reggie replied, "through being taught" (Reggie Questionnaire, June 27). Three others indicated that someone must show them: "Slowly, video tapes with instructor lecturing are good since you can pause and repeat many times" (Natalie Questionnaire, June 27).

Question 5 specifically elicited responses about the instructor's role by asking how an instructor could help them learn during class (see Table 16). Ten students replied s/he should work examples, while two suggested s/he should involve students, and one each replied that s/he should be clear and prepared, go slowly, show different ways to do problems, and create a relaxed classroom environment. Clearly, most students in this class believed a mathematics instructor's role included showing them how to complete procedures successfully, roles similar to those played by teachers in the didactic classes described by Boaler and Greeno (2000). Responses attributing learning to the instructor demonstrate student stances as absolute knowers (Baxter Magolda, 1992).

Mr. Reilly's roles compared to students' expectations. In contrast to student expectations, Mr. Reilly rarely worked examples, and when he did, he integrated them into the development of concepts (e.g. see Appendix H). Students quickly became aware of the difference between this instructor and previous instructors. On the third day of class I overheard two students discussing the class as we waited for Mr. Reilly to arrive and unlock the door, “My precalc-one teacher would walk in and start putting examples on the board and not talk about anything; this is totally different" (Fieldnotes, June 29).

Mr. Reilly lectured in a manner that was different from what students expected: class time included well-planned interactive discussions, with attention to conceptual development and making connections explicit. Boardwork was neat and intentionally placed. Mr. Reilly lectured by facing students, made eye contact, smiled and laughed often. While he stayed at the front of the class, he sometimes leaned his back against the board as he told a story or discussed an application, and moved over to the side when he wanted students to focus on a problem on the board. In addition, he used less formal language, personifying the mathematics (e.g. see Appendix I, lines 22 and 24), and sometimes used the word “ain’t.” These techniques may have helped make the mathematics less formal and remote for students.

Mr. Reilly was energetic, interacting with students and attending to their listening capabilities by foreshadowing important ideas, interweaving stories into the lecture, and pointing out the significance of ideas. He often pointed out important ideas he wanted students to use. For example, as he introduced a problem of finding the trigonometric ratios of the angle formed by the line $y = 2x$ and the x-axis, he suggested, “Keep in mind, little facts, little ideas that float around in trig, keep in mind the size is irrelevant” (Observation, June 29). And, as he introduced proving identities, he wrote, “don’t forget that algebra is still in play” (Fieldnotes, July 12). Such foreshadowing may have made it easier to follow his ideas and make the connections he intended.

After they discussed ideas, Mr. Reilly pointed out their significance. For example, after they found the trigonometric ratios of a 45-degree angle using the Pythagorean Theorem, he added, “And the reason they're such a big deal is this is one of the very, very, very few times where you know the angle and you know the ratio, and they're both nice; almost never happens. Here’s two others” (Observation, June 28). A few minutes later, after deriving the ratios, he expanded on why it was important:

> And again, the reason that’s a big deal is because I can’t do that for forty, I can’t do that for twenty, I can’t do that for fifty. If you ask me what the sine of fifty degrees is, I cannot tell you. I can’t, nobody else can, no box ever can. All you can do is try to get close. And for those bad angles like fifty degrees, that’s where you go to the magic box, because magic boxes have no life and they live for no more than to do stupid, menial tasks …Can I ever get it exact? No. Cuz it goes stinking forever, always changes, no pattern, no repetition, the dumb thing just goes forever, it’s disgusting and horrible. But then most practical people don’t care, four or five digits, usually I’m cool. But the mathematician in me, the mathematician in us, we always kind of look at this one with a little snear. The down side of error is this: if I get angles like thirty, forty-five, and sixty, I don’t have to worry about error because I’m perfectly right and anything I do with that number will continue to be perfectly right. The downside of error isn’t just that you’re wrong, but that error grows. If you’re wrong with one of your numbers in the beginning and you do twenty calculations, there’s a good chance you’re really wrong by the end, okay, really wrong by the end. So when we have to, we’ll use these and when we do, we’ll be really careful. (Observation, June 29)

In addition to highlighting the significance of knowing certain trigonometric ratios, this episode also includes an illustration of how he used language to include students, “the mathematician in us,” in an effort to include them in the community of mathematicians.

Students appreciated the nature and content of the lectures.

> Wow, he has a passion to teach math and it comes across. I actually like coming to this class. … His fascination with what he talks about, you see that passion, and that makes you interested all of a sudden. Little did I know there was all that, I had no clue of all those things in math, you can actually teach the background of where it came from. There's books you can even read. Well nobody's ever done that in any math class I've ever had in my life. It's just this is math, this is what we do... I think it's wonderful, and that he's doing it at this level, and taking the time, because he doesn't have to, he could just give us all this, he could just make it very technical and he doesn't do that, so he obviously knows what works. (Natalie Interview, August 10)

Natalie emphasized the lectures were different from any other mathematics class she had taken. In addition, the body language and behavior of other students indicated they were also engrossed in the lectures. Tim contrasted Mr. Reilly’s style of lecturing with a physics professor he had, “He didn’t need a class because he walked in and started talking to the board as he wrote” (Fieldnotes, July 19). So, while students did not get the clear examples and procedures they expected, the lectures contained elements that helped them value what they were learning.

Mr. Reilly’s role as supportive and available. Mr. Reilly also fostered roles as being very personable and supportive. Each day when class ended, he stayed and sat on a desk at the front of class while students approached him to talk. He made it clear he would gladly talk about mathematics any time he was available and he would be in his office for a couple of hours after class (Observation, June 29), but would also take the time to talk about other things that interested them (Mr. Reilly Interview, August 10; Steve Interview, July 21). Natalie and Steve emphasized this aspect of making himself available were important to them (Natalie Interview, August 10; Steve Interview, July 21). While discussing how to teach responsively to students' ways of knowing, Baxter Magolda (1992) found it was important for instructors to offer opportunities for students to get to know them. She added that absolute knowers wanted teachers to demonstrate helpfulness. Students in this class liked the opportunity to personally connect with Mr. Reilly.

In addition to demonstrating his availability, Mr. Reilly made many supportive comments throughout his lectures. The natures of most of these comments were to encourage students to be persistent and to demonstrate understanding that students might feel overwhelmed.

> Unless you've done trig before, and are pretty good at it, you will be overwhelmed. There is no doubt about that, you'll get overwhelmed at some point in chapter five. If you hang in there and keep trying, you will be okay, I promise you. If you fall, I'll help you up, as long as you try. But it's something you want to be prepared for, is in chapter five there is a lot. (Observation, June 28)

He regularly interjected similar comments throughout his lectures. They demonstrated Mr. Reilly's alliance with his students and his belief they could learn the concepts, but they would need to work hard. He also told students when to expect a topic to be more difficult. I asked Natalie if the comments were helpful to her.

> What helps is that first you go in with an attitude it's going to be very hard, and then it's not that hard… I think that's good to prepare; you have to prepare and then you have your expectations. There's no surprises. (Natalie Interview, August 10)

Perhaps telling students when it was going to be difficult motivated them to work harder and not feel dumb when they did not immediately understand. It was clear Mr. Reilly did not want to use the course to "weed out" students and he believed success was based on persistence rather than ability. "A lot of it is impressions, a lot of it is associations. You know, get away from math as a mark of intelligence, math is an opportunity for failure" (Mr. Reilly Interview, August 10).

Support also included the way he spoke to students during lecture and discussion. After describing previous mathematics instructors who made him "feel dumb" for asking questions, Steve described the current class:

> This one responds in a fairly respectful manner to all questions, which is fairly important I think because by doing that he's earning the respect and relationship of his students. (Steve Interview, July 21)

Steve indicated Mr. Reilly's respectful answers helped maintain a safe environment and made him more approachable. The nature of Mr. Reilly's roles and lectures may have helped students persevere. Steve expressed, "And I feel very strong every time I leave his class and that's one of the things I appreciate. I feel really strong about what he's done" (Steve Interview, July 21). However, Steve struggled to make sense of the mathematics once he was alone but took responsibility for his difficulties since he had always struggled in mathematics and did not spend enough time during this quarter striving to learn it (Steve Interview, July 21).

Mr. Reilly's role during whole-class discussion. Rather than lecture, Mr. Reilly provided many opportunities for students to participate through whole-class discussion. In the following episode, which begins after he extended the definition of sine and cosine from ratios of sides of right triangles to y and x coordinates on the unit circles:

Mr. Reilly: First off, one-twenty in radians?

Julie: Two-pi over three.

Mr. Reilly: Two-pi over three; okay, the sine of one-twenty is going to be the *y*-coordinate of this point.

Tim: Negative one-half.

Mr. Reilly: Careful, the *y*-coordinate; you're a little early.

Tim: Root three over two.

Mr. Reilly: Because the *y*-coordinate is how far up you are, the *y*-coordinate of one-twenty is root three over two, that's the sine. The cosine is the *x*-coordinate of this point; *x* is negative because we're looking to the left, how far to the left? One-half, so the cosine of one-twenty is negative one-half. Okay, all right? Good, so tangent would be?

Student: Negative root three.

Mr. Reilly: Negative root three, okay, cosecant? …[continued with two more]. Negative one over root three, okay, still good? All right, one more then. I'm going to go, oh, let's say we go to this angle, but I'm going to go backwards. I'm going to go clockwise to this angle. In degrees, who is it? [pause],

Jake: Negative one-twenty.

Mr. Reilly: Close, you're within a hundred or so.

Jake: Oh, negative one-fifty.

Mr. Reilly: You can see it from the symmetry, or realize that if I go from here to there, I'm thirty degrees short of one-eighty, that's one-fifty in the negative and remember that ain't just one angle, it's one spot, but it's a whole bunch of angles. So at this position here, I'm at a negative one-fifty. In radians what would that be?

Susan: Negative five-pi over six [she answered very quickly].

Mr. Reilly: Okay, why?

Susan: Because it's negative one-fifty and one-fifty is five pi over six.

Mr. Reilly: It sounds to me like you've done some of this before. I mean, what I'm getting at is if you already know them – know them, then only answer a few. Because what I'm getting at is, how do you think it through while you're waiting to memorize them? I mean, after while you just think it and it's there, and that's good. But while you're waiting for that how would you think through that this is a negative five pi over six?

Natalie: Isn't it five times thirty to get there?

Mr. Reilly: You could do that, five thirty's to get there.

Natalie: And then negative.

Mr. Reilly: Yeah, you're going clockwise so you're automatically negative. What I like, yeah, there's a couple ways to do it. One, I already know that thirty degrees is one-sixth of pi, and I'm one-sixth short of a full pi, so I have five-sixths of pi …There's a lot of different ways to do it, I'm not going to force you to do one. Don't stress; figure out something, because this is too much to memorize, you need something to pull it together. (Observation, June 30)

The goal of this exchange was to have students use the new information about trigonometric values from the unit circle and give students an opportunity to use what they already knew about trigonometric functions, the unit circle, and the coordinate plane to find the values. Mr. Reilly gently let students know when they were wrong by responding "careful" and "close," and when they were correct by repeating the answers. However, by telling students when they were correct or not he accepted mathematical authority.

In the preceding episode, most questions required only short answers and Mr. Reilly expanded on some answers to point out the connections. He asked Susan “why,” but since her reply indicated she had previously memorized the angles, he explained the purpose of the exercise was to find ways to think about the angles using the connections discussed in this class. Natalie responded with a way to think about it, which Mr. Anderson acknowledged, then discussed more ways to think about it, rather than have other students reflect on Natalie’s response. In general, he interpreted and explained mathematical concepts, made connections explicit, and suggested ways to think about the mathematics that he believed would be helpful, a role best described as an intermediate authority (Smith, 1996).

Although Mr. Reilly and his students interacted throughout lectures and students interrupted with questions and comments, Mr. Reilly deliberately maintained social authority throughout class.

> I am a control freak; I gotta drive ... so part of that is premeditated, I'm grabbing control of the class and part of the way I'm grabbing control of the class is louder, more up, more energetic, so you don't have a chance to take control of the class from me… What is a positive way to maintain control? Cheerlead, motivate, because if I do all that, by golly, it's my class, you know which is why, the reform stuff, part of it is philosophical, part of it is I have a hard time giving up control. (Mr. Reilly Interview, August 10)

Mr. Reilly’s teaching was very well-planned and intentional; he had spent many hours thinking about how he knew the mathematics, what connections could be made between concepts, and reading about the history of mathematics and etymology of words (Mr. Reilly Interview, June

29). He carefully motivated each concept and derived formulas, portraying the mathematics as a coherent, connected system of ideas.

Baxter Magolda (1992) suggested that to help students develop more sophisticated ways of knowing, instructors must “situate learning in the students’ own experience” (p. 270). The characteristics pervading Mr. Reilly’s lectures and demeanor such as using language that included students in the community of mathematicians, and fostering personable and supportive relationships with students, situated learning in students’ experiences and supported students’ development of higher ways of knowing (Baxter Magolda, 1992). However, Mr. Reilly maintained roles as mathematical and intermediate authority rather than allow students to test the validity of their own ideas, listen to each other, and develop voice.

The Nature of Social Norms

Attendance was good throughout the quarter: Eighteen students originally enrolled in the class and one dropped within two weeks. Of the nineteen days I observed, there were 15-17 students present fourteen days, 13 students present on one day and 14 present on the remaining days. Students seemed to value coming to this class. Class time consisted of lecture and whole-class discussion; there was no group work or individual seatwork during class, although some students, such as Natalie and Shawna worked together outside of class (Fieldnotes, July 6). Other students also may have worked together outside of class since it was clear some students such as Tim and Jeremy, and Reggie, Nick, and Jake (Fieldnotes, June 28) knew each other before this term. Many students carried on conversations with their neighbors before and after class.

Social norms of whole-class discussion. Whole-class discussion consisted of communication between Mr. Reilly and the students rather than discussion between all members of the classroom community (e.g. see Appendixes H & I). The volume and direction of students’ answers indicated they intended their answers only for Mr. Reilly. He repeated correct answers for the rest of the class to hear, usually reworded to be clearer (e.g. Appendix H, lines 35 - 43) and sometimes appeared to ignore wrong answers (e.g. Observation, June 29). I observed one exception to his repeating correct answers when he remained quiet after several students provided correct answers, giving other students an opportunity to continue thinking (Fieldnotes, July 21). He evaluated correct answers with “good,” and wrong answers with “close” or “no,” and did not ask students to respond to their peers’ ideas. These evaluations may have limited the likelihood that students would listen and reflect on their peers’ ideas (van Zee & Minstrell, 1997) and restricted students’ responses to function univocally (Wertsch & Toma, 1995). Correspondingly, students' answers were usually short and needed further amplification, interpretation, or expansion. There were some exceptions when students provided longer responses (Observation, July 6; Observation, July 13).

Much of the interaction early in the quarter when social norms were first negotiated included closed questions requiring recall as illustrated by the following exchange.

Mr. Reilly: Sine of ninety?

Student: Zero.

Mr. Reilly: Careful.

Several students: One.

Mr. Reilly: Good. See the circle, okay, how high are you? That’s what sine is. So, sine of ninety is one. Cosine?

Ss: Zero.

Mr. Reilly: Zero, good, tangent?

Student: Undefined.

Mr. Reilly: Undefined, okay, as you get more familiar, try to remember things like this, tangent is slope. Okay? What is the slope of a vertical line?

Susan: Zero [OC: this is said so quietly that he cannot hear it].

Student: Undefined.

Mr. Reilly: Undefined, good, so the tangent of ninety degrees makes sense to be undefined, because it doesn't have a slope. Cosecant?

Ss: One.

Mr. Reilly: One, secant?

Ss: Undefined. (Observation, June 30)

There was very little time for all students to think before at least one student volunteered an answer and Mr. Reilly would say whether it was correct or not. This episode was typical in that Mr. Reilly pointed out connections that had already been discussed such as the sine of an angle as the *y*-value on the unit circle and the tangent of an angle as the slope of the line through the point and the origin. Mr. Reilly had previously discussed these concepts; the goal of this discussion was to help students use these connections to become proficient at finding the trigonometric ratios of the angles. In addition, Mr. Reilly usually continued the discussion as long as one person gave a correct answer, as if the entire class had given the correct answer. Although it is likely he did not hear Susan's answer of "zero" in response to the slope of a vertical line, being able to use the connection he intended relies on students' understanding of slopes of lines, which she may not have had.

Purposes of Mr. Reilly's questions. Mr. Reilly often posed questions meant to give students an opportunity to think and make connections. For example, when they were considering multiple measures for coterminal angle:

Mr. Reilly: How big is that angle? How many degrees?

Student: Ninety.

Mr. Reilly: Good, is that the only possible answer?

Student: Two-seventy.

Mr. Reilly: Two-seventy, well two-seventy? Not two-seventy [pause]. Uh, it's things you've done before plus a little kick. Yeah, it's true this could be a ninety-degree angle, that's right, but that's not the only way to do it, you could also do this [draws an arc indicating a four hundred and fifty degree angle], same picture, different idea, see it's not just a static picture like this, it's the motion you took to get there. And if I went from here to there, sure you turn ninety degrees, but I could also do this, puke, you know and go all the way around, but if I do that, then what I've got is ninety degrees plus three hundred and sixty degrees, I've got four hundred and fifty degrees. Or, if you know what, if I really have no social life at all, I could start turning around and go like this, what do you suppose we call that?

Student: Negative two-seventy.

Mr. Reilly: Negative two hundred and seventy degrees because I'm going two hundred and seventy degrees in the opposite direction. Remember, this is three; this is negative three, okay. Counterclockwise is positive, clockwise is negative, so this would be negative two hundred seventy degrees, so I'll ask you again, if I draw you this picture, what do you know about how big that angle is?

Student: It could be anything.

Mr. Reilly: It's almost that bad but not quite that bad. It couldn't be anything but it's definitely more than one thing. Can you describe for me all things it could be?

Student: Divisible by ninety.

Mr. Reilly: No, no, because one-eighty is, so is two-seventy [pause]. This will be a pain in the neck forever [pause]. This angle is ninety degrees plus any multiple of three hundred and sixty. Every three-sixty is a lap, positive multiples of three-sixty are laps counterclockwise, negative multiples of three-sixty are laps clockwise, so anytime I draw that, this angle here, if somebody says, tell me about angle at A, all you know is that angle A is ninety degrees plus some integer multiple of three-sixty. (Observation, June 27)

The purpose of Mr. Reilly's question was to get students to think about the situation and formulate a description of all angles with the same terminal side. While some students may have been thinking, they were not the first to answer and Mr. Reilly responded to the first answers, saying whether they were right or wrong. He continued to rephrase the question but did not give students much time to continue thinking, and finally provided the answer. Since this was the first day of class, this initiated a social norm that he would provide answers when students were not able to quickly come up with a correct answer. Also, when one student responded with a correct answer, it was acknowledged and used as if the whole class had given this answer.

Mr. Reilly's responses to student input. Mr. Reilly interpreted any student response as a signal to continue. On at least two occasions, Mr. Reilly asked if there were any questions and Susan quickly responded "no" (Fieldnotes, July 27; Observation, July 13) for the entire class. However, once when she responded for the whole class and Mr. Reilly continued, Mark interrupted to indicate he did have a question (Observation, July 13). Although Mr. Reilly lectured at a quick pace, Susan seemed to try to speed it up more with quick answers that often appeared to be guesses or by speaking for the whole class. Mr. Reilly discouraged this behavior early in the term by pointing out that she should not speak up so quickly but give others a chance, “I mean, what I’m getting at is if you already know them – know them, then only answer a few” (Observation, June 30). So, Mr. Reilly seemed aware of the impact and had a strategy to handle it.

Although many of Mr. Reilly’s questions required only short-answer responses and were conversational in nature, Mr. Reilly occasionally followed students’ answers with “how’d you get that?” as in this exchange where they were finding the period of $y = \sin\left(3x - \frac{\pi}{2}\right)$ by setting up the inequality $0 \le 3x - \frac{\pi}{2} \le 2\pi$ and solving for x:

> Mr. Reilly: Two pi over three, excellent, how'd you get it?
>
> Reggie: The difference between them is four pi over six and I just reduced it.
>
> Mr. Reilly [speaking to entire class]: It's that easy and it's that hard. What's the period? How far is it from here to there [indicating final inequality] but unfortunately, you're used to living in a world like this [writes $1 < x < 6$]. How long is that interval?
>
> Student: Five.
>
> Mr. Reilly: Five, good, see you’re used to those numbers and if somebody says how long is this period, you go, eh, five, cool, one to six is five. There is something about putting little Greek letters in that messes everybody up. Get used to it, accept it, then deal with it. How long is a period? It’s from here to there, [points to Reggie] subtract, reduce, two pi over three, okay. So, two pi over three. (Observation, July 5)

While acknowledging Reggie’s contribution, Mr. Reilly expanded on it, making a connection to students’ previous understanding. He attributed the idea to Reggie, but also rephrased it. By repeating and expanding correct ideas, he maintained his role as authority. This teacher role does not align with Baxter Magolda's (1992) recommendations; she argued that in order to foster more complex ways of knowing, instructors must suppress their own authority and encourage students to provide ideas for their peers to consider (Baxter Magolda, 1992).

At other times, Mr. Reilly provided reasons for students' answers without asking them for an explanation, such as in this episode when he had just graphed $y = \sin x$,

> Mr. Reilly: Mark this off in units of pi, forward and backwards. Oh, can you see the range of the function from here?
>
> S: Negative one to one.
>
> Mr. Reilly: Negative one to one, good, because I'll never go lower than negative one and I'll never go higher than one. When *x* is zero, I'm here, from zero to pi over two, sine goes from zero to one [he's referring to the unit circle as he graphs]. So, this is one and this is negative one, I go from zero to one. When you go from pi over two to pi, then I go from one to zero, okay. Here, third quadrant, pi to three pi over two, sine goes from zero to negative one. And the fourth quadrant it comes back again, and if you go the other way, you get the same shape here. (Observation, June 29)

In general, Mr. Reilly asked questions to give students opportunities to think and to make connections. However, he explained the answers for the class; he stated that his rationale for this approach was that he could tell students far more connections than they could discover during class (Mr. Reilly Interview, August 10).

Students appeared to appreciate opportunities to ask and answer questions throughout class. Near the end of the term, students interrupted with questions far more often than in the beginning of the term (Fieldnotes, July 26). This was a result of Mr. Reilly's helpful responses when students interrupted (e.g. Appendix H, lines 37-65), and because they had gotten to know him. Students had also gotten to know others who sat near them since there was much discussion between them before and after class (Fieldnotes, June 30). During class, however, communication remained between Mr. Reilly and the students rather than between all members of the class.

Use of class time. There were no opportunities for individual or group work when I observed. Mr. Reilly had tried using group work in a previous class. He described a paired developmental/ study-skills course he co-taught with a study-skills instructor (Mr. Reilly Interview, June 29). The other instructor insisted on group work during class and Mr. Reilly agreed to include it as long as they used the time set aside for her portion of the class; he did not have the time to spare. The final grades for the top and middle students in that class were similar to the usual grades for the course, but the bottom group of students had grades about .5 higher on a 4.0 scale than usual (Mr. Reilly Interview, June 29).

Mr. Reilly's goal was to provide a structure by illuminating concepts, key connections, and important points. Natalie appeared to accept this purpose of lectures when I asked if she had changed the way she studied compared to previous math classes:

> [I'm] reading more, the book, going back over and, it's funny, everything he says pops in my head, maybe two or three days later. And I got that, I think three days into the course, I realized it because, I realized, boy, listen to what he says, because he's not just saying things to say things. (Natalie Interview, August 10)

Mr. Reilly did not expect students to understand immediately, but only after they worked outside of class. Natalie realized she would not necessarily understand the connections made during the lectures until after she worked on her own. Her comments indicated that she strived to understand the mathematics outside of class. So, while students interacted with Mr. Reilly during his lectures, he allotted no time for individuals and groups to make connections, discoveries, or "do mathematics" during class, but believed students should continue striving to understand outside of class.

While the content of lectures may have challenged students' beliefs about the nature of mathematics as concepts rather than procedures, the social norms did not challenge their beliefs about their roles as receivers of knowledge. The social norms supported Mr. Reilly's role as intermediate authority in the classroom rather than provide expectations for each student to engage in mathematical practices during class. Unlike students in the class described by Rasmussen et al. (2003) whose beliefs about their roles changed because of their instructor's explicit attention to social norms and roles, Mr. Reilly's responses allowed students to maintain roles consistent with their previous experiences and expectations.

How Mr. Reilly Portrayed Mathematics

This section contains evidence of the way Mr. Reilly portrayed mathematics to students through stories about the history of mathematics, the usefulness of the mathematics they were learning, and the language he used to discuss mathematics. I also show that these factors affected students' ways of knowing.

Mathematics grounded in human history. Throughout the term, Mr. Reilly portrayed mathematics as concepts and tools devised by real people trying to solve practical problems. This representation included describing real life applications of mathematics and providing the origins of newly introduced words. For example, on the first day of class:

> Mr. Reilly: Trig is very, very old, we've been doing trigonometry for three, four, five thousand years, which means the people who do trig are practical people and they are theoretical people. It's kind of weird when we get those groups of people mixing and

talking about something. And here's a good an example of that, any body know what that word means? [underlines the word trigonometry] Anybody got any Greek to them?

Tim: Trig, triangle.

Mr. Reilly: Trig's for triangle, good, ometry? [Looks around, pausing] Measure. You're taking a course on measuring triangles, is what you're doing, okay, but you know how many people you can scare with that? You don't scare anybody. What'd you do today? Well, we spent an hour measuring triangles. No, it doesn't work. Because it's old, because it has a heritage, because it has a history, we tend to use old words, things like Greek, trigonometry; that means measuring triangles. Two, over the years, mathematics has achieved the status of intellectual elitism, so we like that, so we intimidate people. Part of the reason I say trigonometry is because it scares you, because it makes me smarter. If I can do trig, I'm a smart person, and you'll see that. There are things we do in trig, there are things we do in math because it's the right idea, there are things we do in math because we've always done it that way, there are things we do in math because it scares people and makes us feel smart. (Observation, June 27)

Rather than introducing trigonometry as knowledge that exists, Mr. Reilly depicted it as developed by different people with different purposes. He did not imply that a person must be smart to be successful in mathematics, but portrayed some aspects as attempts to look smart and alienate others.

Mr. Reilly told stories while developing concepts. The stories were not long and each had a purpose, usually to give historical context to the mathematics. As he introduced trigonometry with triangles, he developed the progression of ideas by using similar triangles, then using ratios of triangle side lengths. He next recounted a story of Thales measuring the height of a pyramid.

Mr. Reilly: The first example of I actually know of somebody doing this is an old Greek guy by the name of Thales, lived about five hundred b.c. He studied math, he's a little bit earlier than Pythagoras, and at some point, he takes the package tour of Egypt, and apparently when he was in Egypt, apparently he was talking himself up pretty good, I'm a great mathematician, I'm really smart, I'm Greek you know. And the Egyptians called him on it. The Egyptians said fine, if you're so blasted smart, here you go, there's a pyramid, find the height, okay. The reason this is a significant problem is, the simplest way to go get the height is to just go measure it. Go get a tape measure, go up from top to bottom, find out how long it is. What's the down side of that?

S: Pyramid is huge.

Mr. Reilly: Pyramid is huge, and it's made out of rock. In order to measure its height directly, you're going to have to dig a hole from the top of the rock to the bottom of the rock, and it's just not practical. So Thales uses our little triangle trick and comes at it indirectly, and this is what he does. He says all right, let's wait 'til we get a nice sunny day, which in Egypt ain't that hard to do. So you wait until the sun is in position where you get a nice, nice, nice shadow. Now, notice that when I get a shadow of the pyramid here, what I've actually formed is a triangle. See it? The height to the ground forms a nice little right angle. By the way, I get another side of the triangle along the ground, and the third side of the triangle goes to the top of the pyramid. So there's one triangle. Thales said all you have to do is this. Go get a stick, I don't care how big the stick is, just go get a stick of some fixed height. So, let's say he gets a stick six feet tall, there it is, walk up and down until the shadow from the top of the stick matches the tip of the shadow from the top of the pyramid, and oh, hold it straight up because you want a nice little right angle here. And notice what you have. You have one triangle here, this triangle I'm going to name ABC, and I'll write it over here (see Figure 2). ABC, with a right angle. Here, I've got a second triangle, ADE, I'll put that over here, ADE. (Observation, June 28)

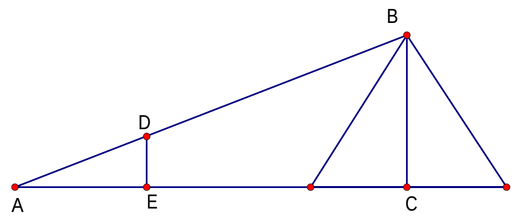

Figure 2. Heights of a pyramid.

As usual, Mr. Reilly faced students as he talked, drew a picture on the board, smiled, and modulated his voice so the story captivated listeners. The story depicted mathematics as developed by clever people to solve real problems.

In his discussion of the history of mathematics, Mr. Reilly mentioned that mathematicians chose certain mathematical conventions for convenience and the conventions continued to be used after the convenience was obsolete. For example, while explaining the choice of 360˚ for a circle:

> And if you're doing math on your fingers, that's a nice number, sixty is good. Three-sixty is six sixties, everybody goes into three-sixty: two does, three does, four does, five does, six does; almost everybody goes into three-sixty. Well, were these people thinking about calculators? No. Great, we have calculators now; do you think we're going to change it? No, cause I ain't changing the books. It's old math, the reasons for doing it are all gone, but it's still going to be there. (Observation, June 27)

In a similar vein, Mr. Reilly also discussed reasons for rationalizing denominators (Fieldnotes, June 30) and why degrees are decomposed into minutes and seconds (Fieldnotes, June 27). The combined effects of these explanations was to portray conventions as human choices and ground mathematics in human history.

A related technique was to discuss ambiguous notation:

> Be careful, the sine of alpha plus beta is not the sine of alpha plus the sine of beta. This is not a multiplication, it's a function, … beware of anything that makes you think it's like a product. Again, it's not your fault; we did that. When we decided not to put the parentheses there, we set you up for that, so watch for it. (Observation, June 29)

With his warning to be cautious, he also indicated that the problem was with the notation and not the students. He again placed the blame on the mathematicians who developed the notation when he warned students that the notation for inverse trigonometric functions did not have anything to do with reciprocals (Observation, June 30; Fieldnotes, July 18). These warnings may have influenced students' beliefs that their difficulties in mathematics were inherent in the notation and not in students' abilities. It also added to the sense that Mr. Reilly was sensitive towards their struggle to learn.

In addition, he spoke to students about the nature of mathematics, "Don't think of math as this eternal unchanging entity completely divorced from opinion. Swokowski [text author] has one perspective, Cole [second text author] has another, and I have mine" (Observation, July 6). From this statement, it appeared that only experts such as mathematicians, mathematics instructors and text authors could have differing opinions. However, that different mathematicians may have different opinions introduced the idea of mathematical knowledge as uncertain which challenged absolute knowing (Baxter Magolda, 1992).

Mathematics as real life. Mr. Reilly often paused to tell students how the mathematics they were learning was used in real life. For example, after graphing $y = \sin x$, he described its uses:

> Unintentional benefit, anything in the real world that does this, vibrates, cycles, repeats, uh, earthquakes, music, radios, heartbeats, anything that does this, this is what we use. In fact, we're so good at this, this is what a.m. radio is made out of, these. This is what f.m. radio is made out of, these. If you've ever heard a computer speak with a human voice, it is using some variation of this. Kind of a neat unintentional consequence, sine as a graph has a wave and that has lots of neat applications. (Observation, June 29)

He leaned his back against the board, facing students and making eye contact while moving his hands to indicate a wave. Students appeared to enjoy the stories and explanations of real life applications. On his student questionnaire, completed at the beginning of the first day of class, Jeremy indicated that providing real life applications could help him learn (Jeremy Questionnaire, June 27). Both Natalie and Steve indicated in their interviews that the stories and real life connections made the mathematics much more interesting. Steve said,

> I appreciate hugely the fact that he takes the time to relate some of the stories that he's found over the course of his twenty years of teaching through reading. And they're really interesting to me. They're way more interesting than crunching numbers but yet they have to do with why we're crunching numbers. It gives you some background. To me that's the next best thing from coming out here and measuring the tree first hand, is the fact that he's doing that, he's telling stories about where these came from, or even little side notes of what you can do with what you've just learned. Oh, this is how this person took this tool and then measured, you know, an entire country with it, or figured out the distance to

> Mars. It's awesome …As I told him before, what he's doing for me is a god-send, as far as I'm concerned. From all the time that I've put into math classes that either I passed or I failed, and I hated them, or I wasn't interested, or I saw no application, or was bored out of my mind, and I knew that there was supposed to be something there, but I just couldn't apply it to anything. Now I can start to apply it to something. (Steve Interview, July 21)

Steve valued knowing that the mathematics he was learning could be used to solve real life problems. This was clearly a different perspective of mathematics than he had before this class and was caused by Mr. Reilly's portrayal of mathematics. By discussing the origin of words and telling stories that illuminated the historical development of the mathematics they studied, Mr. Reilly made it more accessible and tangible to students, and demystified it. The effect was to situate learning in students' experiences by inviting them to share in the experiences, which encourages more complex ways of knowing (Baxter Magolda, 1992).

As a result, Steve showed evidence of crossing between absolute knowing and transitional knowing. His questionnaire responses emphasized a need for clear, step-by-step procedures and heavy dependence on the instructor, demonstrating a predisposition of absolute knowing. However, his appreciation of Mr. Reilly's attempt to encourage understanding showed evidence of transitional knowing. During our interview I asked Steve what he was going to do to get ready for the next test, he answered,

> [Laughed] All the things that have been taught to me in my previous math classes, which are probably all the things that I shouldn't be doing, because I've been kind of bred to do things a certain way. And I think he's trying to break that mold that I've been put in and I see that in him. But I'm not sure how to at this point unless I took the teacher home with me. Um, I know that I should, I know that I have to, in order to come into the real world

> mathematics that he's trying to teach. But it's difficult, it's difficult when you've been, like I said, I've taken so many math classes and it's always the same thing, just go home and crunch numbers and just get upset, confused, angry, and a lot of times you need to sedate yourself just to sit there. I mean, I honestly get A's, almost straight A's through all my other classes, except for math. (Steve Interview, July 21)

While Steve recognized and appreciated that Mr. Reilly expected him to learn differently in this class, he did not have strategies for learning in new ways. However, he approached me on the next class day I observed and made a point of telling me he tried a new study technique over the weekend: he rented the videos that go with the book and watched them with paper and pencil in hand, stopping and replaying as necessary (Fieldnotes, July 26). While this showed he was willing to try something new, he still relied on an authority. Thus, students may need support to study and learn in new ways while their current ways of knowing are contradicted.

Mathematics as truth. Although Mr. Reilly portrayed conventions and the development of mathematics as human inventions and choices, he also believed deeper truths existed in mathematics:

> I think there's two perspectives to math. One is, math is something that belongs to truth and we discover math as we go along. Another is, math is something that is man-made and artificial that is a scaffolding that is used to approximate what you see out there. I like the former. The former is what is math? Math is what is. So the Pythagorean Theorem is true, and the only truths that are discovered by every culture, at every time, in every place, to me are essentially mathematical. So, I got that from Euclid,...Every time I do math I'm looking for a bigger truth... The notation is just the scaffold,,...The concept is that bigger, deeper, philosophical truth,… In terms of doing that [regarding incorporating

real life applications and stories], that was one of my goals and it was slow,… reading those books, prepping lessons. (Mr. Reilly Interview, August 10)

Mr. Reilly had examined his own beliefs about the nature of mathematics and his beliefs influenced the way he told stories and his reason for telling the stories. His passion for mathematics grew out of reading Euclid's Elements and other mathematical books (Mr. Reilly Interview, June 29). He tried to portray the same image of mathematics to his students; finding the history and creating lectures that included history and real life examples was intentional and took a lot of time.

In summary, Mr. Reilly portrayed mathematics through stories about the development of mathematics and real-life uses, and used language that personified mathematical objects, encouraging access to mathematics and situating learning in students' experiences. Students valued these aspects and perceived this class as different from previous mathematics classes.

Summary of Research Question One: Nature and Development of Community

The majority of students entered this class as absolute knowers, believing mathematical knowledge to be certain. They believed the instructor had the knowledge and they obtained it from him; his role was to do many examples and explain clearly. The roles and social norms did not provide students opportunities to change this view since they were rarely asked for their ideas, were not expected to listen to each other, and Mr. Reilly sanctioned correct answers and explained, maintaining his roles as mathematical and intermediate authority. It follows that the nature of social norms and roles did not support students' development of voice and opportunities for contextual ways of knowing in which students' used their own sense-making to validate mathematics.

However, Mr. Reilly's portrayal of mathematics through history and his comments about opinions in mathematics challenged student beliefs about the certainty of mathematical knowledge. Similarly, his emphasis on understanding concepts encouraged transitional knowing while his use of real life applications situated learning in their experiences. While those factors challenged absolute knowing, Mr. Reilly responded to the needs of absolute and transitional knowers by demonstrating a caring, supportive attitude and providing opportunities for students to know him. Responding to students' needs in their current ways of knowing is essential for "heightening students' interest in learning, strengthening their investment in that process, creating comfortable learning atmospheres, and developing relationships that foster understanding" (Baxter Magolda, 1992, p. 268). By confirming and challenging students' current ways of knowing, Mr. Reilly encouraged more complex ways of knowing (Baxter Magolda, 1992).

Research Question Two: Interactions Related to Mathematics

The following sections contain a discussion of the nature of classroom interactions as they related to mathematics, especially the nature of communication, sociomathematical norms, and of doing and learning mathematics.

The Nature of Communication

Communication in this class was through lecture and whole-class discussion. Table 17 shows the percent of time spent on each of the communication types described by Brendefur and Frykholm (2000) for the first five days of class.

Uni-directional communication. The nature of uni-directional communication in Mr. Reilly's class varied from straight lecture, to conversational closed questions, such as "Anybody know what that word means?" (Observation, June 27). This type of question initiated a norm that students should join in the conversation. Uni-directional communication also included recall questions or fill-in-the-blank statements to see if students could use meanings recently introduced. The following episode provides an example; Mr. Reilly had just introduced meanings for the sine, cosine, and tangent of an angle from a right triangle.

> Mr. Reilly: So if I go back over here, clear a spot, and grab my three, four, five, and call this alpha. From this triangle, sine of alpha would be?
>
> S: Four over,
>
> Mr. Reilly: Four over,
>
> S: Five.
>
> Mr. Reilly: Four over five; standing here at the angle, opposite is four, hypotenuse is five. So the sine of alpha here is four-fifths. Make sense? Next one up, is called cosine, sine's buddy, abbreviated cos, cosine is adjacent over hypotenuse. So, on this triangle here, the cosine of alpha would be? [pause] The adjacent side is,
>
> S: Three over five.
>
> Mr. Reilly: Three over five, good. (Observation, June 28)

The episode contains closed questions to see if students could use the definitions just introduced. It also allowed students who wanted to be more active a chance to participate, while giving Mr. Reilly an opportunity to assess whether or not students could apply the information just given and slowed the pace of the lecture. When one student replied correctly, Mr. Reilly did not assume all students understood but explained why it was the correct answer. However, he also had very little wait-time and after asking for the cosine of alpha and a slight pause, asked a simpler question rather than asking a question to find out what students were thinking (Herbel-Eisenmann & Breyfogle, 2005; Cazden 1988/2001).

Some uni-directional communication could best be described as funneling (Wood, 1998). After finding the angular velocity of the Earth, they address the next problem in the text (Swokowski & Cole, 2002):

Mr. Reilly: Okay, forty-two [referring to the problem number in the text], the equatorial radius of the earth is about three thousand, nine hundred and sixty-three point three miles, uh, find the linear speed. Find the linear speed at the equator [laughs].

S: That's an evil laugh.

Mr. Reilly: Oh, yeah, actually, it's part of teacher training; you've got to have an evil laugh. All right, so what am I going to do?

S: It doesn't give you anything.

Mr. Reilly: Actually, it gives you everything you need. What do you have?

Tim: Numbers, letters.

Mr. Reilly: I got this [points to result from #41].

Tim: Which equals seven hundred and [inaudible].

Mr. Reilly: It's bugs, that's all it is, bugs. Here's angular speed; here's the units I have, what are the units I want? [By “bugs” he is referring to the problem he used to introduce the distinction between linear and angular speeds.]

S: Miles.

Mr. Reilly: It's bugs, what are the units I want? What is linear speed measured in?

S: Inches.

Mr. Reilly: Inches? Miles? Feet? Length? So I need miles, so I want miles per second. So, what do I multiply by; what are the units? [pause] I cannot tell you how good this trick is. What are the units I multiply by? Who dies? [The “trick” is dimensional analysis.]

S: Miles over radians.

Mr. Reilly: Radians die, who lives? Who rises?

S: Miles.

Mr. Reilly: Good, miles per radians, give me an equivalence. (Observation, June 28)

This episode of solving a homework problem started with a question aimed at eliciting a correct procedure, but a student responded that they did not have enough information. When Mr. Reilly argued and asked what information they did have, Tim gave a flippant answer, to which Mr. Reilly responded by funneling with shorter closed questions, and finally provided the answer he wanted. The nature of this episode was affected by the social norm that once a student gave an answer, Mr. Reilly answered it, by his role of providing correct answers when students did not.

Mr. Reilly sometimes followed students’ answers with “why?” (e.g. Observation, July 6). In the following episode, he was finishing an explanation of a homework exercise that asked students to write tangent in terms of sine only. He asked why, but students’ responses were not helpful in promoting reflective communication, so he answered it himself.

Mr. Reilly: Since sine squared plus cosine squared is equal to one, that means cosine squared is equal to one minus sine squared; and cosine squared is, well at least at first, plus or minus the square root of one minus sine squared. However, that one goes away [erases the minus sign] why?

S: It can't be negative.

Mr. Reilly: Cuz it can't be negative, and why can't it be a negative?

S: Because it's a radical.

Mr. Reilly: Actually, it's outside the radical, though, so that's not the issue… What is cosine to you right now? [pause] …It's a ratio; it's one length divided by another length, and if you take one length, divide it by another length, can you get a negative ratio?

S: No.

Mr. Reilly: No, a positive divided by a positive is positive, and for right now, that's positive. (Observation, June 29)

Mr. Reilly emphasized that there was a reason why the expression could not be negative, but the students' answer "it can't be negative" did not answer "Why?" so Mr. Reilly repeated his question more clearly. However, a student said "it's a radical," apparently associating radicals with "can't be negative" so after mathematically refuting their answer, Mr. Reilly answered his own question, referring back to the only meaning of sine they had seen so far in class. This episode became an instance of uni-directional communication and declined in cognitive demand when Mr. Reilly allowed very little wait-time and then answered his own question. Henningsen and Stein (1997) implicated teacher taking over and lack of time as reasons for decline in cognitive demand. In addition, the classroom social norms and roles encouraged this type of communication since students could give short answers and Mr. Reilly responded by saying whether it was right or wrong until either a student gave the correct answer or he provided it.

Through some uni-directional communication Mr. Reilly determined what students did not know and adapted his instruction. In the following episode, he was able to determine and address students' current understanding of identities:

Mr. Reilly: But before I get into that, a quick review of what an identity is. An identity is an equation that is not meant to solve, it's just meant to demonstrate. Here's an example, what's another way to say two times *x* plus *y*?

S: Two *x* plus two *y*.

Mr. Reilly: Good, two *x* plus two *y*. Notice when I do that, I write that as an equality. That's not supposed to be an equation to solve, that's not the purpose of it; it's not a question. It's supposed to be a demonstration: see, this is the same as this. Likewise, something like this, *x* plus three, squared, is something like this, *x* squared plus six *x* plus nine. The whole point is not to solve it, in fact, if you did try to solve this, what would the answer be? [pause] Who works? [From fieldnotes: Mr. Reilly smiles and moves away from board; some students shake their heads no as if there is no value that "works"].

S: Irrationals.

Mr. Reilly: Yeah, irrationals work, who else? [pause] And notice how many people are jumping up and down saying call on me, call on me, call on me. How many numbers work in this equation? [pause] Let me put it a different way: who doesn't work?

S: Zero?

Mr. Reilly: Eh, zero works. If you put zero in here, that's three squared, that's nine. If we,

S: Negative three.

Mr. Reilly: Negative three works, if you put a negative three in here, you get zero, if you put negative three here, you get nine minus eighteen plus nine, that's zero. Okay, cool. Who doesn't work? [pause] Well, I'm glad I brought this up. You've been doing this for two years by the way. Nobody fails; that's actually the point. It doesn't matter who *x* is, that's always true. You can make *x* three, four, the square root of twenty nine, you can make *x* pi, if you put that number here and put that number over here, it's the same thing. That's the value. And the reason you like identities so much is because they allow you to change questions. [Gives an example of solving an equation] …I have a whole handful of identities, I know that these two are the same, so any time I get one of these, I can stick one of these in there. So, I can change the form of the question to this. And now it's a much nicer question that still has the same answer. That's the value of an identity, I can change the way it looks to make my life better. Okay? (Observation, June 29)

Mr. Reilly continued emphasizing the meaning of an identity while considering each trigonometric identity. After deriving the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, they checked its truth-value for $\theta = 30^{\circ}$, $\theta = 60^{\circ}$, and $\theta = 45^{\circ}$. In this instance because no student demonstrated an understanding of identities, he spent time discussing their meaning. In this instance he used uni-directional communication, specifically closed questions, to determine students' current understanding and adapt his instruction.

Contributive communication. Communication that focused on making connections or discoveries, or prompted students to make sense of the mathematics, in addition to problem-solving episodes could often be characterized as contributive. In the following episode, Mr. Reilly posed the problem intending for students to make a connection:

> Mr. Reilly: Keep in mind, little facts, little ideas that float around in trig, keep in mind the size is irrelevant. Here's an example of that, on the old *x*-*y*-plane, here's an *x*-axis, here's the *y*-axis. I'm going to draw the line $y = 2x$, which goes through the origin. Notice that when the line *y* equals two *x* passes through the origin it makes an angle with the *x*-axis, okay, right here. I'm going to call that angle theta. Question: Find sine theta, cosine theta, and tangent theta. [From the time he finishes this sentence until a student answers is 18 seconds] You notice what I'm not asking you for; I'm not asking you for theta, we don't need it. What I want is the sine of theta. Ideas?
>
> S: Well, you can draw a line down, anywhere.

Mr. Reilly: Good, excellent. The problem with finding it right away is that you don't have a triangle. If you had a triangle, you'd do the opposite over hypotenuse, and that would give you sine of theta, so I'm going to make a triangle. And you're right, the idea, I'm going to take some point on the line, and drop, to make a right triangle, what point?

S: Doesn't matter.

Mr. Reilly: Doesn't matter, any point you want. So I'm going to pick any point on the line. I'll pick this one [labels (2, 4)]. If I pick the point two, four, I naturally make a right triangle, because, since the coordinates are two and four, this length here is two and this length is four. What about the hypotenuse? [pause 5 seconds] I keep saying something about that.

S: Four. [Mr. Reilly waits, looking around and looking at the student who said four.]

S: [a different student] Pythagorean.

Mr. Reilly: Yeah, Pythagorean [smiles at student who said four]. The hypotenuse is going to be the square root of four squared plus two squared, which is the square root of twenty. Oh, and by the way, I won't get hung up about simplifying radicals, as long as you don't snitch on me, you don't go telling your next teacher, Reilly didn't make me clean up radicals…[questions to find side lengths of triangle]…Notice anything about tan? [pause 4 seconds] I'm going to put this down here. For that line, tan of theta is opposite over adjacent, is four over two, is two. The tangent turns out to be two, but actually two is important for another reason [pause 6 seconds].

S: It's opposite over adjacent.

Mr. Reilly: What's the slope of that line? Its slope? Opposite over adjacent is rise over run; it's the same thing. Tangent is slope. One of the things we're going to do this quarter

> that's kind of interesting is, instead of trig being over here and algebra being over there, no, we're bringing trig to algebra. They're going to merge, they're going to talk to each other, sometimes they'll actually get along. The tan of the angle is the slope of the line, again because opposite over adjacent is rise over run. (Observation, June 29)

This episode was slightly over four minutes. While solving this problem, Mr. Reilly had specific ideas he wanted to highlight: trigonometric ratios did not depend on the size of the triangle but on the ratio of the lengths of the sides, and the tangent of an angle is the slope of the line. One student offered an idea “draw a line down anywhere,” using the idea that size is irrelevant. However, near the end of the episode, a student gave an answer repeating what Mr. Reilly had already said, “It’s opposite over adjacent.” While Mr. Reilly introduced this problem to allow students to make a connection between tangent and slope, a social norm of this class was that if a student answered, Mr. Reilly evaluated the response and then gave the correct answer. This social norm constrained opportunities for students to make the connection.

Some contributive communication arose from students’ questions and allowed Mr. Reilly to understand what students were thinking (see Appendix H). Mr. Reilly intended to engage and challenge students’ thinking with the initial posing of the problem, warned students on the necessity of thinking, and asked questions on the number of solutions and why. The episode started and ended with uni-directional communication (Lines 1-18 and 40-67), but contributive communication arose in this episode when Tim shared his ideas in the form of a question (Line 30) that showed he had been trying to make sense of the solutions, which contradicted his understanding of the sine of an angle as its height on the *y*-axis. Mr. Reilly initially misunderstood Tim’s question (Lines 31-32) and Tim clarified what he meant (Lines 33-34). Mr. Reilly appreciated Tim’s contribution and reworded it for the rest of the class before demonstrating that the solutions worked. This episode provided an opportunity for reflective communication since Mr. Reilly could have asked students to explore the question. However, because the roles and social norms emphasized Mr. Reilly's role as intermediate authority, reflective communication was not likely. Mr. Reilly emphasized that students should ask similar questions and strive to make sense when they were studying outside of class.

The above episode contained some elements of instructive communication in that Mr. Reilly posed the problem to encourage reflection (Brendefur & Frykholm, 2000), and Tim's question illuminated his understanding of these solutions and the contradiction it presented with his current understanding of the sine of an angle, modifying instruction. It contained an element of reflective communication, since once Mr. Reilly understood what Tim was saying he rephrased Tim's question so other students could also consider it, making it an object of discussion (Brendefur & Frykholm, 2000). However, the communication then returned to uni-directional communication as Mr. Reilly asked, then answered his own questions.

In summary, the discourse in this class consisted of uni-directional and contributive communication. Uni-directional communication was often interactive, allowing Mr. Reilly opportunities to assess students' use of the concepts and adapt instruction. In addition, interactive lecture provided opportunities for students to be more active if they wanted. Although Mr. Reilly presented problems to give students opportunities to think, make sense of the mathematics, and make connections, roles and social norms constrained opportunities for higher communication types and positioned students much like the students Boaler (1999) observed in the traditional classes at Amber Hill. Boaler observed that teachers told students whenever they were stuck and did not foster peer relationships which would support different ways of knowing.

The Nature of Doing and Learning Mathematics

This section begins with a discussion of students' preconceptions on learning mathematics, then presents evidence of the way Mr. Reilly modeled and explicitly discussed ideas about doing and learning mathematics, then addresses other elements of doing and learning mathematics that emerged as sociomathematical norms.

Students' initial beliefs about mathematics. Two questions on the Student Questionnaire (Appendix D) targeted students' beliefs about the nature of doing and learning mathematics. Their responses to the question, *How do you best learn math?* indicated they learned mathematics by following examples and by practicing step-by-step procedures (see Table 15). Eight students responded that practice was important, six students responded that being shown examples with step-by-step procedures helped, and five students indicated that the instructor must teach them. None of the students mentioned working with others, discussion, or exploring to this question, demonstrating they had a predominantly procedural approach to mathematics and were absolute knowers.

A second question asked if students thought memorizing steps and formulas in mathematics was important (see Tables 18 & 19). A majority (9 students) thought it was important for at least some things, while those who said no said they would be able to look up what they needed. Students arrived in this class already believing that memorizing was important but later indicated that this course was different. Tim said this class was "really different from previous courses" because previous courses focused on memorization, "memorizing is shoved down your throat" (Fieldnotes, July 12). And, on his questionnaire, Steve said he best learned mathematics by being given a specific step-by-step method for attacking problems, and then practicing. He also said that memorizing steps was very important (Steve Questionnaire, June

27). However, after a few days of class he approached me and said the answers on his questionnaire would be different now (Fieldnotes, July 6). Later, in an interview he said, "The situation with trig functions is a little different because you can't put a step-by-step process to it. …. I don't know if I could learn the way I'm learning now with the trig functions and put it to the algebra functions" (Steve Interview, July 21). He appeared to believe that the nature of the mathematics he was learning now was different from what he had learned in the past and was the reason why the focus in this class was on concepts instead of procedures. Thus, while students arrived with beliefs that mathematics required memorization much like the students from the didactic classes discussed by Boaler and Greeno (2000), some changed their beliefs.

Explicit messages about doing mathematics. Throughout the course, Mr. Reilly emphasized understanding and being able to use concepts rather than memorizing procedures. He did not show students procedures without first illustrating the concepts, and then connected the procedures to the concepts. On the first day of class, Mr. Reilly explained the nature of learning the content of this course:

> The other thing that's different is, precalc one, almost entirely analytical, it's almost entirely mechanical, it's almost all algebra. Okay, precalc two is that but it's also conceptual, it's also visual, it's also geometric. … most problems have multiple techniques. And your idea is you want to understand the tapestry, you want to understand what trig is, and then when you have a problem, you realize there are multiple ways to get from here to where you want to be, but you understand them well enough where you make that choice based on quality. (Observation, June 27)

Mr. Reilly had a clear map of how the concepts of trigonometry fit together and taught new ideas by connecting them to older ideas, carefully developing concepts from previously discussed concepts and deriving formulas they used. We discussed his conception of teaching:

> Yeah, I like the fact that I have a map. Uh, to me it's more a, more of a mantra to myself that everything needs a conceptual driver… You don't do things just to do them. Because once you get that bigger picture, not only do you anticipate results, but also you see it in other classes, in other disciplines, you see it in other places. Because you see the bigger picture, so you don't have to memorize stuff. Um, and I think that's the development, was first, everything should be there for a reason and some things should be questioned…But, the next step from there is that if everything is going to fit into the big picture, that means everything should have a conceptual driver... It should flow. (Mr. Reilly Interview, August 10)

Mr. Reilly focused on concepts throughout each class period and used many opportunities to suggest the same to students. For example, when teaching transformations of trigonometric functions, he pointed out that the book had given them formulas but added that it was, "a waste of brain space, I wouldn't memorize anything, understand it" (Observation, July 6). Instructors' emphasis on understanding are appreciated by transitional knowers (Baxter Magolda, 1992).

Mr. Reilly often made comments regarding the nature of doing mathematics: "If you're not thinking, you're not doing mathematics" (Observation, July 5), and explicitly discouraged memorizing formulas, insisting students make sense of what they were doing. On the second day of class, after asking questions to see if they were familiar with arcs:

> Mr. Reilly: And what I'm curious about is, let's see how much you got, here's a circle of radius r, okay, I want to know how long that is. [Indicated the arc subtending an angle θ.]

S: Theta *r* squared.

Mr. Reilly: Nope, and unfortunately you're working on what you remember rather than what you see, but you're on a different page. What I want you to do is I want you to think it through. All the way around would be?

S: [very quietly] Two pi *r*.

Mr. Reilly: Okay, if I want to go all the way around, from here all the way, the whole, one lap around this track would be,

S: Two pi *r*.

Mr. Reilly: Two pi *r*, okay, so I want to know, how big that would be. And what I want to do is, I'm going to give you two formulas that actually you won't see again for a year and a half if you're in math. … I don't want you to memorize them. Because if you memorize them, you'll forget them the day after the last test. I want you to see where they come from and where they come from is this: this is a fraction of the whole, right? It's not the whole lap, it's one fraction of the lap, what's the fraction? What part of the circle am I talking about there?

S: Theta over three-sixty.

Mr. Reilly: Okay, theta over three-sixty. Theta over three-sixty; this is the fraction of the whole. Okay, does everybody see that? Okay. That was a really, really good idea, that was a really, really good idea but he's just going to get absolutely punished for it. Um, but I'm going to go ahead and do it anyways to see why he gets punished and to see what we're going to do about it. So, if the length, the arc length is going to be that fraction of the full circumference. Theta to three-sixty of two pi *r*, okay. And I could make that, I could make that my arc length formula, it would be fine. I'm not going to though. Why'd you pick three-sixty?

S: It's the whole.

Mr. Reilly: Two pi is also the whole. If I go in radians, two pi radians is a whole circle, why did you go to degrees instead of radians?

S: The angle's in degrees.

Mr. Reilly: That's because you did what we all did. Why do you measure in feet and inches instead of meters? Because you're better at it. It's harder, but you've been doing it for so long you're better at it. … Theta out of,

S: Two pi.

Mr. Reilly: Two pi. Does this make this a tad bit nicer? Yeah, it does. See it? It goes bye-bye now, you do this, bingo, that's the arc length, that's it, okay?... Now I want to do the same thing for area. Here, same thing, let's do the whole thing again. The area of the whole circle, if I go all the way around, is pi *r* squared. Same thing, though, I want to get the area of this sector. I got a radius of *r* and an angle of theta, I'm going to use exactly the same thought process. Fraction of the whole, if I'm only looking at theta, what part of the whole do I have? [pause] I'll give you a hint, it's the same answer you had last time. [pause] The whole circle is how big?

S: Three-sixty.

Mr. Reilly: Not three-sixty, no, it's forgivable, but I don't like three-sixty.

S: Two-pi.

Mr. Reilly: Two-pi, good, and how much of that two pi do I have right here?

(Observation, June 28)

Mr. Reilly emphasized being able to derive, not memorize the formula. He bluntly said, “Nope” to the first answer, then added that he wanted them to “think it through,” and specifically addressed a weakness of memorizing. After funneling students through the derivation, he provided students an opportunity to use his reasoning on the next derivation. However, the student who answered “three-sixty” clearly did not understand his earlier point. Thus, while he emphasized reasoning and provided a specific way to reason through the derivation, there is evidence that students did not follow the reasoning.

Mr. Reilly explicitly discussed metacognitive practices, reflecting on or asking students to reflect on both processes and results. For example, throughout verifying an identity, he asked students to give reasons why they selected certain procedures and emphasized they should have mathematical reasons (Fieldnotes, July 12). After verifying the identity, he asked “Could we have done it better? Let's do it again and see if we can do it better.” The next day, after verifying another identity using students’ suggestions, he said, "Look at and think about what you did," and pointed out where they had reversed a step on the first verification (Fieldnotes, July 13). His emphases on thinking about what they were doing and why directly contradicted some students’ beliefs that mathematics consisted of memorizing steps.

The nature of solving problems. Students contributed during whole-class problem-solving episodes (see Appendix I). Throughout the interaction, Mr. Reilly emphasized problem-solving techniques of understanding the problem, having a plan, and reflecting on the answer. His stance at the beginning of the episode encouraged contributive communication, however he did not pursue suggestions that would be unproductive. While students offered several suggestions of sides and angles he could find, he said they needed to have a plan. As he used some of their suggestions to write a plan, he labeled the sides or angles without calculating, and pointed out that while they could find certain lengths and angles, many of the results would not get them any closer to finding a solution to the problem. Finally, he reflected on the process by discussing the helpfulness of drawing a picture and Reggie's suggestion of drawing a line. The goal of this activity was to explicitly discuss processes of mathematical problem-solving.

Learning mathematics includes exploring. Mr. Reilly modeled and suggested that learning mathematics requires exploring. He used the word "play" several times in this context,

> But a lot of the name of the game in chapter five: go play, go play. These problems are good because they make you live in two or three different worlds [different representations]. And also too, even though that going to the graph didn't work, you now learned something; don't go to the graph on a problem like this, it didn't help. (Observation, July 30)

The message to students was that they could learn from using several representations and that they could also learn when they were not able to solve a problem.

In our last interview, Mr. Reilly discussed his reasons for not incorporating more explorations into class time:

> For one thing, discovery learning took ten thousand years. No, I ain't going to do that. True discovery learning means going down the dead end… Um, reform, in high school, if it were done right, I could see by slowing it down to that one-third, one-half pace, you could take every other day and just get nowhere, just play, you know, go do this and connect some dots… But I go back to three hours per credit per week, so that's fifteen hours a week. If you want me to do group work, you want me to do discovery, give me fifteen hours a week. That's my philosophy, you give me fifteen hours a week with these kids, then you bet I'll take them down discovery, I'll figure out group work, But you

know, you took two-thirds of it from me, so my job is to motivate the other two. (Mr. Reilly Interview, August 10)

Mr. Reilly saw time as a serious constraint; he had far too many concepts to illuminate and connections to make for students to make them all during class, so he would make sure he covered them knowing that students might not understand until later.

> I don't think they have that many epiphanies in class. I think the best thing I can do is provide the continuity, provide the math, this is the way down the road, …[I] want them to go home and do it, absolutely; that's where it happens. (Mr. Reilly Interview, August 10)

Mr. Reilly believed most student learning occurred outside of class when students were able to think at their own pace and study in ways they found helpful.

Near the end of the quarter Natalie expressed a similar view:

> I was doing all these exercises yesterday and I'm going, I'm getting them but I don't think I quite know what I'm doing. It's working, but, and that bugs me because I want to go back 'til I get it. I have done that, I've stayed all day on one problem. It was pathetic. It was one he went through in class, it was right in my book, but I refused to look at the book, that note. I wanted to get it on my own, my own way, all day, almost the whole day. But boy that process allowed me to get it and I won't forget it. (Natalie Interview, August 10)

Her assertion that she wanted to solve the problem in her own way and did not want to look at Mr. Reilly's solution evidenced she wanted to think for herself, an indication of independent knowing (Baxter Magolda, 1992). However, another statement during the same interview, regarding going to the Mathematics Lab for help, "it's just if there's a teacher there, I'll ask him, an instructor, just because he knows" (Natalie Interview) demonstrated she believed mathematical knowledge to be certain and owned by authorities, and indicated absolute knowing (Baxter Magolda).

Consistent with his belief that students should work outside of class, Mr. Reilly presented problems that they did not solve in class but were expected to do on their own. For example, he introduced a problem from *Trigonometry for the Practical Man* (1946), and did not solve it in class but told students there were many ways to do it and it would be on the test. He added that he wanted to see how they planned as they problem-solved and emphasized that they could not use any procedures they memorized earlier unless they had reasoning to back it up. In the hall before class Tim showed me the work he did trying to solve the problem; the page contained 12 – 14 drawings, although there were no labeled points or words anywhere on the page (Fieldnotes, July 21). The exam question asked, "Describe how to find the distance from A to B. Remember that you can only measure angles and distances on your side of the river. Draw all necessary triangles and justify your steps" (Mr. Reilly Chapter 6 Test, July 25). The accompanying picture showed two points on one side of a river and a person on the opposite side, but included no numbers or angles. Students were expected to describe a plan for finding the distance.

In an instance arising from a student question about a formula for $\tan\frac{u}{2}$, Mr. Reilly used questioning to derive a formula (Fieldnotes, July 19). In the last step, the minus sign disappeared from in front of the radical sign, and he asked why. "How can they ignore that?" No one answered and he did not answer it, but he said he would put the question as a bonus on their next exam. Thus, Mr. Reilly modeled and discussed problem solving but expected students to practice solving problems outside of class and provided opportunities and incentives to do so. Susan showed evidence of finding the answer to this question since she expressed a belief that the bonus points would bring up her exam grade (Fieldnotes, July 27).

Mr. Reilly sometimes used students' ideas to model mathematical practices and reflect on results. For example, after they derived a formula for $\cos 2\alpha$, he presented the problem of finding the cosine of fifteen degrees. When asked what they wanted alpha to be, one student said "fifteen," another said "thirty," then someone repeated "thirty." So, he used $\alpha = 30$ until they arrived at a dead end. Then they replaced alpha with fifteen degrees and ended up with a square root preceded by plus or minus: "Do I really think that the cosine of fifteen degrees has two signs?" (Fieldnotes, July 18). In this episode he explicitly modeled going down a dead end so students could see what happened and reflected on the result. That is, he modeled and explicitly discussed the sense-making and "playing" he expected students to do outside of class.

Emphasis on multiple representations. Mr. Reilly used and discussed the importance of using multiple representations to solve problems. While solving $\sin x = \frac{1}{2}$, $\sin x < \frac{1}{2}$, and $\sin x > \frac{1}{2}$ on $[-2\pi, 2\pi]$, he drew a unit circle and graphed $y = \sin x$:

> Mr. Reilly: Thirty, okay. I've got thirty degrees here, I've got thirty degrees here, they want me in radians, but I'll take care of that when I get there. I kind of like degrees better. So, first picture is this, on the unit circle you're at two different places. On the wave, another way to think about it is, if the sine is equal to one-half, this is where the sine is equal to one. Sine is equal to one-half, half way up, so your answers are here, here, here, here, [he draws a dotted line at $y = \frac{1}{2}$, intersecting the curve in four places]. And which kind of reinforces the first picture. If you're taking two laps on the circle, these two points are four solutions and on here, you actually get to see the four. Okay, your choice, name

the four, how do you want to do it? [pause] Who are they? [pause] And let me put it this way, would you rather think on the circle or think on the wave?

Ss: Wave, circle. [Natalie and Shawna said “wave” first, and then at least two of the male students said “circle.”]

Mr. Reilly: Okay, so let's do it this way; who's this? [points to one of the points intersecting the dotted line on the curve] See if you're on the wave, that's the way you're thinking, who's this? [Some students speak softly.]

Shawna: Pi over six.

Mr. Reilly: Pi over six, good. Where'd that come from?

Natalie: A half.

Mr. Reilly: Yeah, good, memorization, maybe a little bit of this [points to circle], and the bigger point is there are three pictures to trig: triangle, circle, wave, triangle, circle, wave, how many of them am I using right now?

Susan: All of them.

Mr. Reilly: All of them. That's trig. There are triangles, inside of circles, which create waves. And to really do trig, and to get comfortable with it, you want to get comfortable with the whole thing. (Observation, July 5)

Consistent with *Principles and Standards for School Mathematics* (NCTM, 2000), Mr. Reilly valued mathematical practices such as exploring, reflecting on processes, problem solving, multiple solutions and multiple representations and discussed these practices explicitly with his class. He emphasized the need for them to engage in these practices when they studied but did not provide opportunities for students to engage in these practices during class.

Sociomathematical norms. Some aspects of learning and doing mathematics were less explicit and could best be described as sociomathematical norms since they were social norms related to the evaluation of mathematical activity (Yackel & Cobb, 1996).

While Mr. Reilly valued understanding concepts and connections, one salient sociomathematical norm in this classroom was that acceptable answers used the reasoning or meaning that Mr. Reilly had in mind when he asked the question. This episode illustrates his very clear sense of how he wanted to develop certain concepts. I also noticed that the lengths he gave could not make a triangle; I do not know if any students realized this.

> Mr. Reilly: Everything in trig basically starts with one simple trick. Here it is, here's a triangle, I'm going to make that seven long, I'm going to make this five long, and I'm going to make that two long. And now I'm going to extend it, blow it up. For your generation, I'm going to zoom in. In another words, I'm going to make it bigger, but I'm not going to change the shape, so it's going to look like this. Same exact shape, just bigger. Uh, go with me, same shape. And, I'll tell you part of it. I will tell you this side here is twenty-one long, how big is *x*?
>
> S: Six.
>
> Mr. Reilly: Good, you know trig, that is trig, okay. How'd you get it? How'd you reason that through?
>
> S: Seven, three times bigger.
>
> Mr. Reilly: Good, that's it, hang on to ideas like that, from seven to twenty-one ratio, you notice that the triangle on the right is three times bigger than the one on the left. From that you assumed that that ratio extends to all pieces of the triangle. So *x* must be three times bigger than two, is six. The only thing you did wrong is you did it with common

sense, which you should know is just illegal in math. So now I'm going to make you fix it. Give me an equation that solves to be x equals six.

S: [a couple of students answer] Twenty-one over [she stops]; seven over twenty-one is equal to two over x.

Mr. Reilly: Good, seven is to twenty-one as two is to x. Okay, let's see if it works, cross multiply, seven times x is equal to forty-two, yeah, it works. Give me another one. That's actually not the one I wanted. [pause] It's kind of subtle, but that's not quite the equation I was looking for. Here, I'll give you the one I was hoping for. [Writes $\frac{7}{2}=\frac{21}{x}$.] That's the one I wanted. It's not your fault though. If you knew, it's probably because you already did trig. Um, but first, notice it works. Seven times x equals two times twenty-one, there's forty-two, and what do you know, I got six. Um, so apparently the only thing that was different between those two was the vision. What's different in what I'm seeing here versus here?

S: Each side belongs to one triangle.

Mr. Reilly: Perfect, perfect. Each fraction belongs to one triangle; that's what makes this one different. This belongs to the triangle on the left; this belongs to the triangle on the right. (Observation, June 28)

The distinction between the equations was important since Mr. Reilly used the idea that once they have a ratio, they no longer need a second triangle. However, his statement that the student's answer was "not the one I wanted," implied his questions were usually intended to solicit specific best answers rather than students' ideas. This sociomathematical norm allowed for more instructor control over the mathematical ideas that could be afforded significance. Mr. Reilly had a highly connected and rich understanding of the mathematics and he wanted students to have the same understanding, but did not expect them to have it right away (Fieldnotes, July 13; Mr. Reilly Interview, August 10).

A related sociomathematical norm was that Mr. Reilly provided meanings or connections and then questioned students to facilitate their use. For example, while finding trigonometric values of angles around the unit circle, he asked for the tangent of -150°:

> S: One over root three.
>
> Mr. Reilly: One over root-three, intuitively speaking, why should it be positive?
>
> S: They're both negative.
>
> Mr. Reilly: Yeah, but visually, visually, why, from the circle, why should tangent of negative one-fifty be positive?
>
> S: Slope.
>
> Mr. Reilly: Look at the slope, yep that line has a positive slope, and since tangent is slope, since this thing has positive slope, anybody in the third quadrant is going to have a positive tangent, because everybody in the third quadrant has a positive slope. It'll come from negative over negative. (Observation, June 30)

Mr. Reilly wanted students to use the connection between tangent and slope. However, the answer, “they’re both negative” is also a good way to think about the tangent and could be considered intuitive or visual since any point in the third quadrant has negative *x*-and *y*-coordinates. Mr. Reilly may have realized that when he added, “It’ll come from negative over negative,” but he asked the question to facilitate students’ use of the connection between tangent and slope, so his response initially focused on the connection he had in mind when he asked the question.

However, students appeared to heed Mr. Reilly's regular comments that they needed to strive to understand and make sense of their answers. For example, although Tim interacted early in the quarter by giving memorized answers (e.g. Observation, June 28), and pushing Mr. Reilly to tell them how to do the problems (e.g. Observation, June 28), in the episode on July 13 given in Appendix H, he was clearly trying to make sense of the answers using meanings Mr. Reilly had focused on several times, that the sine of an angle is the *y*-coordinate on the unit circle (e.g. Observation, June 30; Observation, July 5).

Another student showed evidence of striving to make sense during class. Although at the beginning of the term Julie responded to whether she found memorizing important, "Yes, it is the most important –because math is rules to follow to complete a problem," (Julie Questionnaire, June 27) in the following episode, she was trying to make sense of the diagram on the board.

> Mr. Reilly: Find the inverse sine of one-half, okay, what they mean by that is this; what they mean by that is find a theta, and I'm going to say, find the theta for which sine of theta is equal to one-half. Okay, do trig backwards. What angle has a sine of one-half? [pause 4 seconds] this is how well you want to know the picture.
>
> Susan: Forty-five [quietly; Mr. Reilly did not appear to hear].
>
> S: Thirty [nearer the front].
>
> Mr. Reilly: Thirty, good, I'll go thirty degrees. And here, I'll redraw the picture because this is definitely something you should see. The visual picture you have goes something like this; here's your unit circle, okay, again this is one, this is one, if sine is one-half, that means you are one-half up, your height is one-half. You said thirty degrees, thirty degrees is good, because if I draw a thirty-degree angle, then at this point on the circle I am one-half up and that works, cool. What's the problem? [pause 4 seconds.]

Julie: Well that's a third isn't it and not a half?

Mr. Reilly: One-third of the angle, but one-half of the axis. See, you're thinking this way, that's one-third of the way along the circle,

Julie: Right.

Mr. Reilly: But it's only halfway up. Sine's how high up you are and the angle is how far along the circle you are [pause]. You don't look convinced. See, if I say the sine of thirty degrees is one-half, okay, that means, or it might be even better, the way you're talking, it might be better to say the sine of pi over six is one-half. Angles are measured along the circle, so if I say pi over six radians, I literally mean that distance,

Julie: That arc right there?

Mr. Reilly: That arc length,

Julie: Okay.

Mr. Reilly: That's what a radian is.

Julie: Okay.

Mr. Reilly: The sine of the angle is that height, the *y*-coordinate, okay, that's what sine geometrically means is how tall are you? And what I've done is, basically asked, if your height is one-half what's the angle that got you there? And the answer I got was that for a height of one-half, pi over six or thirty degrees will get you there. So I'm really tempted to do this, the inverse sine of one-half is equal to thirty degrees. But, coming back to my issue, what's wrong with that?

S: It's a function so it can't work in degrees.

Mr. Reilly: What's that?

S: It's a function so it can't work in degrees.

Mr. Reilly: Oh, okay.

S: Is that what you meant?

Mr. Reilly: Nope, but I'll take it though. Or thirty degrees, out of respect. Um, but I still have issues.

S: It could be one-fifty. [This is the answer he was looking for.] (Observation, June 30)

This episode was just under three and a half minutes long. The day before, Mr. Reilly drew 30º-60º-90º and 45º-45º-90º triangles with hypotenuse lengths of one on a rectangular coordinate system and interactively developed the idea of sine and cosine as the *y*-and-*x*-coordinates on the unit circle, respectively. That meaning was not yet taken-as-shared although Mr. Reilly spoke as though it were, because Julie thought an angle whose sine is one-half would be halfway between the axes along the arc, an idea probably influenced by the symmetry of the circle. While this episode occurred early in the term, Mr. Reilly had already emphasized that they needed to try to understand and Julie's question demonstrates she was trying to make sense. However, her cognitive conflict was not resolved by addressing what she saw but by appealing back to the previous day's established meanings. In addition, other students who contributed were clearly trying to follow Mr. Reilly's thinking and guess what answer he wanted. Thus, sense-making to Mr. Reilly meant being able to use the concepts and connections he introduced rather than students' resolving their perceived inconsistencies. In addition, rather than ask Julie to explain further, he assumed he knew what she was thinking.

Another sociomathematical norm was that Mr. Reilly did the mathematics. Since the social norms and roles provided that all discussion included him, and he evaluated answers, repeating correct ones and responding "no" to wrong answers (for exceptions see Fieldnotes, July 18, and Fieldnotes, July 6), he stayed at the center of every discussion and controlled its direction. Appendix I provides an example of a problem-solving episode where he chose from among students' suggestions, effectively maintaining control of the direction and pace of the solution process. This sociomathematical norm was strongly supported by his belief that there was not enough time for individual and group work during class and students should spend their out-of-class study time solving problems. However, as discussed earlier, he presented problems that he did not solve in class but he expected students to be able to solve on exams.

A final sociomathematical norm related to the theme that procedures must be supported by mathematical reasons. When a student answered a question procedurally, Mr. Reilly sometimes asked them for their reasoning:

> Mr. Reilly: How many minutes, I'll make it really nice, in three degrees? Nice question, I mean feel free to jump right to the answer.
>
> S: A hundred and eighty.
>
> Mr. Reilly: A hundred and eighty, good; how'd you do that?
>
> S: Multiplied.
>
> Mr. Reilly: Good, now why'd you multiply by sixty instead of dividing by sixty? and what told you that? I mean simple questions.
>
> S: Three groups of sixty. (Observation, June 27)

As discussed earlier, this was consistent with his explicit message that students should understand the concepts and have mathematical reasons for their procedures (Observation, June 27). Students occasionally invoked the textbook (Fieldnote, July 6), or a previous instructor (Observation, July 5) as their reasons but Mr. Reilly pointed back to mathematical reasons, sometimes asking students for the reason, but more often supplying the reason. Simon and Blume (1996) also found students who justified mathematics by referring to previous teachers and the textbook as authorities and confronted them. Although he pointed to mathematical reasons, when Mr. Reilly explained the reasons he supported absolute knowing since absolute knowers believe it is the instructor's responsibility to explain and to explain in a way they can understand. "Maintaining an awareness of the knowledge level of the class helps keep explanations within reach of students' understanding" (Baxter Magolda, 1992, p. 232). However, silencing his own authority and expecting students to provide the mathematical reasons would have challenged absolute knowing.

Summary of Research Question Two

Mr. Reilly was a clear and organized instructor who had planned each lesson carefully developing concepts by building on previously developed ideas. He modeled and discussed a disposition to do mathematics (NCTM, 1991) by solving problems in more than one way, discussing strategies, using multiple representations, and emphasizing the need to explore. He mentioned that completing the homework was not enough; if students knew the concepts well enough, they could do any problems using those concepts (Fieldnotes, June 27), suggesting that students should strive for a deep and connected understanding. Students such as Julie, Steve, Jeremy, and Tim behaved in ways consistent with more complex ways of knowing than they evidenced in their questionnaires.

However, interaction in this class did not promote, and at times constrained, students' own sense-making or individual agency to make sense of the mathematics since social norms and roles allowed students to offer short answers and guesses while Mr. Reilly did the mathematics. Social norms and roles constrained the level of communication to uni-directional and contributive, and sociomathematical norms implied that mathematics should make sense but that making sense was limited to the ideas provided by Mr. Reilly, rather than understanding constructed by students. Mr. Reilly's roles as mathematical and intermediate authority conformed to expectations of absolute knowers, who believed it was the instructor's role to provide knowledge and theirs to accept it (Baxter Magolda, 1992; Brew, 2001).

Chapter Five: Findings in Mr. Anderson's Class

Background

Three to eight students in Mr. Anderson's first-quarter precalculus class waited each morning for the classroom to open as students and instructors walked through the wide hall lined with trapezoidal benches. Thomas sat on a bench working on his homework or reading a book each day I arrived and he was willing to chat with me most mornings, but declined when I asked him for an interview. I was also able to have informal conversations before class with Daniel, Carol, Sarah, Kevin, and Anthony.

Mr. Anderson arrived promptly each day at 7:30 a.m. and unlocked the room. The classroom was very large and contained about sixty desks spread out in rows with ample space on all sides. Greg, a Russian immigrant, sat in the far back left corner by himself. Kathy and Anthony knew each other before this class and sat next to each other, seeming to enjoy working together whenever they had the chance. Two students, Sarah and Carol, already had bachelor's degrees in education and took this class as part of a secondary endorsement in mathematics. They worked closely together all quarter although they did not know each other or sit together on the first day. Carol's teaching certificate was in special education but she taught algebra at a local high school for three years before No Child Left Behind legislation required her to have a mathematics endorsement to teach mathematics (Fieldnotes, August 11). During the school year before this study she taught eighth-grade mathematics. Sarah worked as a substitute elementary teacher.

I recognized both Kevin and Janet from my own institution; they also recognized me. Both had taken this course at least once before. Sheila was rarely in class so I have very little data on her. Thomas and Carol were present every day I observed, although Carol sometimes walked in late. Kevin, Daniel, Sam, Sarah, Greg, and Brian each only missed a couple of days. Anthony and Kathy each missed several days, but usually when one was not present, the other one was. Janet started missing class after the first couple of weeks and then missed several days. Because of where I sat and who talked the loudest, I heard Carol, Sarah, Daniel, and Kenny the most often, although I rarely heard student-talk during group work.

Research Question One: The Nature and Development of Community

The following sections describe student expectations of their roles and Mr. Anderson's roles. I also describe how participants negotiated and maintained roles and social norms, and the development of relationships within the classroom. Finally, I discuss how expectations of roles and social norms revealed students' ways of knowing.

Negotiating Mr. Anderson's Roles

Students' expectations of Mr. Anderson's roles. At the beginning of the quarter, most students indicated through their responses to the Student Questionnaires (Appendix D) that their mathematics instructor's role was to show them procedures while explaining clearly. This idea threaded through responses to several of the questions. To the question, *How do you best learn mathematics?* the majority indicated they learned mathematics best when someone showed them how or explained (see Table 20). When asked specifically about things the teacher could do during class to help them learn, ten of the thirteen students indicated that the teacher should do examples or explain clearly or in detail (see Table 21). Daniel's response, "Make sure they've answered my question so that I understand it" (Daniel Questionnaire, June 27) displayed the common idea that students' understanding was a result of the instructor's explanations.

In response to what types of input they usually offered during class discussions, six students wrote they would ask questions, while only two responded they would offer other types of responses (see Table 22). “Yes, ask questions about things I did not understand” (Sheila Questionnaire, June 27) was a typical response. These examples show that students expected to be able to ask questions and receive clear answers from Mr. Anderson. Responses to whether listening to other students’ questions or explanations helped them learn also indicated that students’ roles were to ask questions and the teacher’s role was to explain (see Table 23). Nine of the thirteen responses specifically indicated students liked to hear other students’ questions: “Listening to their questions and hearing the teacher’s explanation helps,” (Kathy Questionnaire, June 27). These responses indicate that although students may have appreciated the ideas introduced by other students’ questions, they wanted clear explanations from the teacher.

Mr. Anderson’s conception of his roles. Mr. Anderson’s ideas about his role were consistent with students’ ideas. When asked if he thought of himself as a traditional or reform teacher, he said he had heard the words but did not really have a clear idea of what they meant. His image of a traditional instructor included one who did all the talking, so he did not think he was traditional. But he also thought that most mathematics instructors who lectured were more interactive than what he considered to be “traditional” because they provided opportunities for students to participate (Mr. Anderson Interview, August 16).

He developed his conception of his roles as an instructor based on his experiences of how his students preferred to learn:

> A lot of times they don’t think they can, but they could try to read it, see it on the board, or somehow they got some, they see it, they read about it, how to do it, or they have a resource where it is shown to them. And then, practice, basically, as practical learning, that’s what I would think. And then as far as teaching goes, well, then I try to mimic that

even though they could read it out of the book, but most of them don't like that for some reason. (Mr. Anderson Interview, July 5)

While Mr. Anderson believed students could read and learn from the textbook, he realized that many did not like reading the book and preferred to have it explained, so he accommodated this preference. His use of the word "shown" indicates he believed students learn mathematics by seeing someone else perform procedures, pointing out connections or explaining meanings and definitions, and then practicing what they saw, able to use the meanings and definitions that were explained. Both the students' and Mr. Anderson's ideas of how students learn mathematics are strikingly similar to Smith's (1996) characterization of common views of learning mathematics that lead to teaching by telling: "Students learn by listening to teachers' demonstrations, attending carefully to their modeling actions, and practicing steps in the procedures until they can complete them without substantial effort. Solving problems is a matter of recalling and applying the procedure appropriate for a given problem type" (Smith, 1996, p. 391).

Mr. Anderson's response to students' expectations. Mr. Anderson behaved consistently with his students' expectations about the role of a mathematics teacher. Throughout each class period he presented concepts, examples, and problems, encouraged students to ask questions, and responded with thorough explanations. He provided time for students to work problems in class, listened to students, provided feedback, explained what they said they did not understand, and facilitated whole-group discussions. He showed his willingness to explain early on the first day when reviewing how to solve several types of equations:

Mr. Anderson: Next one [writes $x^2 - 8x + 20 = 0$]. Okay, let's try completing the square on this one. It may or may not factor, but let's just try completing the square on it. Anybody remember?

S: Yeah, *x* squared plus [sic] eight *x* plus something equals negative twenty.

Mr. Anderson: Okay, right, you move the number over when you're completing the square, so you have *x*-squared plus [sic] eight *x*, take the twenty and move it over, and we need to add something in here so that it's a perfect square. You have to do one other step too, you have to double-check that your leading coefficient is one. If it was three, you'd have to divide everything by three and you'd get fractions. All right, so we have to figure out what we need to add to both sides here.

S: Take half of eight and square it.

Mr. Anderson: Half of eight and square it? Negative four squared is sixteen, so we want to add sixteen in right here; so that means we should add sixteen to the other side? Since we have an equal sign, we have to keep it balanced. So, now this is a perfect square [indicating left side of equation]; that's why we chose the sixteen. Now, this actually does factor, [writes the binomial squared]. There's actually a trick here if you remember it. So, [pointing to -8] we took half of this squared to get sixteen, but if you just paused halfway, when you took half of the eight, that's what's going to end up down here. So it's a little trick, I almost do this first [pointing to -4] before you come up [pointing to 16]. You know what's going to be in here, before you even write the sixteen. Okay, negative twenty plus sixteen is negative four. Okay, so we can take the square root of both sides. Take the square root of this, we're just going to get *x* minus four. When we take the square root of negative four we need to put something in front.

S: *i*.

Mr. Anderson: We'll get an *i* in the end because of the negative.

S: Plus or minus.

Mr. Anderson: Plus or minus. The *i* comes from the negative, the plus or minus comes from *we* took the square root. Now, maybe we should pause for just a second [he walks over to a clean spot on the board]. Why do you need that? Everybody just look at this *x*-squared equal to nine. Now, I know that we all know that three works there, but there's another number that works too.

S: Negative three.

Mr. Anderson: Negative three works as well, so that's kind of why you need that plus or minus when you take the square root. So the plus or minus comes from taking the square root…[finishes problem]. Four plus two *i* is one answer, four minus two *i* is the other answer. (Observation, June 27)

This episode occurred on the first day of class, which consisted of a quick review of solving many types of equations covered in prerequisite courses. The first student answer was a statement of the result of an action with no reference to the action and no reason for it. Mr. Anderson responded to the answer by agreeing and rephrasing it to clarify that the student's response meant that the 20 should be moved to the other side of the equal sign. His statement, "we have to figure out what we need to add to both sides…" suggested he wanted more than just the number; he wanted to focus on how they could determine what number to add. In response, a student indicated a process for finding the answer rather than just an answer. Additionally, Mr. Anderson addressed "why?" twice in the above episode, showing that he thought explanations were important, but that it was his role to explain. He continued to ask questions, repeat and expand on correct answers, and moved to another part of the board to explain why they needed positive and negative roots to the quadratic equation, a concept he expected them not to understand fully.

Mr. Anderson described the general format for each lesson he taught: introduce concepts followed by easier in-class practice, then assign homework which contained more challenging exercises and problems (Mr. Anderson Interview, August 9). The next day he answered homework questions fully. Students most likely expected this approach because it typifies U. S. mathematics teaching (Stigler and Hiebert, 1999). Homework did not count in students' grades; Mr. Anderson did not collect it or check to see that students had attempted it, so some students may not have tried any of the exercises. However, several students usually had questions, and because he answered them completely with student input, answering homework questions at the beginning of the class period regularly took at least half an hour. This strategy provided students who attempted the homework a night or more to think about the new concepts and try the exercises and problems.

Mr. Anderson's response to homework questions. See Appendix J for an example of a response to a homework question. Although Daniel stated he did not know what was being asked, Mr. Anderson did not probe further to find out what Daniel had tried, making this episode typical in that Mr. Anderson did not ask the student about the source of their confusion, (exceptions were noted on July 5 and July 19). The day before this episode Mr. Anderson had introduced composition of functions but had not presented an example similar to this one. Although the idea of using a point on the graph to be able to determine a function input and output had been emphasized several times earlier in the quarter, students did not seem to recognize this idea would be helpful in this case. After Mr. Anderson included the problem in whole-class discussion and refocused their attention on the meaning of the notation in terms of the graph (Lines 11-35), Daniel was satisfied he could finish the problems (Line 38). However, other students wanted to see more examples, so Mr. Anderson demonstrated how to do the exercises by asking questions and eventually completed all of the exercises in class.

Supporting students' preferred ways of learning. Mr. Anderson's role as explainer and his openness to students' questions accommodated students' preferred ways of learning, and conveyed a high degree of support. He encouraged students to ask questions by the length of wait-time he allowed after he asked if there were any questions. Usually the wait-time ended with a student question. For example, on July 27, he waited 43 seconds after asking if there were any more questions and the wait ended when Sam asked for a specific problem in the homework (Observation, July 27). Wait times near this length were not unusual in this class. In a study in college classes, Karabenick and Sharma (1994) suggested that students' perceptions of teacher support for their questioning affected students' motivation and the likelihood that they would formulate questions; support included providing opportunities for students to ask questions such as extended wait times and providing high-quality answers.

In addition to their questionnaire responses indicating a preference for teacher explanations, students did not have to read the textbook; on his questionnaire, Kevin explained he needed to be shown how to do the mathematics since he could not read the book (Kevin Questionnaire, June 27). Only Thomas responded that he read the textbook (Thomas Questionnaire, June 27). In her interview, Sarah said she looked through the section to make sure he covered everything, but she tried to make her notes thorough enough so she did not need the book.

> I get enough from him, but I think I just kind of skim over it in case there's something he missed. Um, I think he does a very good job of going through the sections, but sometimes, uh, this book is more difficult to do on your own, so looking at it, you know,

> as [Carol], the gal who sits next to me, said, it looks kind of foreign if you just look at it, where when he explains it then a light bulb kind of goes off. (Sarah interview, July 13)

Sarah and Carol preferred Mr. Anderson's explanations to reading the textbook since Mr. Anderson's explanations were easier to understand. Daniel also did not like the textbook, but because it did not contain examples for each type of problem in the exercises. Because of this, he referred to the textbook as "tricky" (Observation, July 5). Students preferred a teacher who was open to questions and willing to interpret and thoroughly explain the mathematics, clearly indicating their positions as absolute knowers (Baxter Magolda, 1992; Brews, 2001).

While his usual response was to explain, Mr. Anderson occasionally provided opportunities for students to continue thinking during class discussions and especially during seatwork. For example, while discussing symmetry, he wrote the equation $x^2 + (y-3)^2 = 9$ on the board:

> Mr. Anderson: So *x* squared plus *y* minus three squared is nine [pause 6 seconds]. Maybe let's do this one differently; let's graph this one. What is this?
>
> Daniel: Parabola.
>
> Sarah: Circle.
>
> Mr. Anderson: I got parabola and a circle [pause 7 seconds]. Do you want to,
>
> S: Circle.
>
> Sarah: You've got a zero.
>
> Mr. Anderson: Circle, Yeah it's a circle, both of them are squared, you've got a zero here. If we didn't have one of the variables squared then it would be a parabola, but since both of them are squared, we have a circle. So where's our center? (Observation, June 30)

Mr. Anderson paused and waited after hearing two different answers, allowing the rest of the class time to consider (a similar instance was noted on July 7). While two students answered after the pause, Mr. Anderson explained why it was a circle and not a parabola, acknowledging Sarah's contribution as meaning that the center of the circle had an *x*-value of zero. While waiting gave students the opportunity to continue thinking and to offer justification, Mr. Anderson's role included providing a final correct answer.

Although Mr. Anderson demonstrated his willingness to explain during whole-class discussions, he was less likely to explain and more likely to ask questions while talking to individual students about their work. In the following exchange, Carol was working on solving: "The sum of the squares of two consecutive even integers is 1252. Find the integers" (Stewart et al., 2002, p. 71). The following excerpt from their conversation illustrates Mr. Anderson's questioning one-on-one:

> Mr. Anderson: Okay, now this is squaring the same thing twice; it's not, you need to square one number, and then the next number, so if this is like twelve or fourteen, what is the next number? Okay, right, if this is fourteen, this is sixteen, if this is twenty, this would be? Okay, in all of those cases, how many more is this one? Two more, so if this is *x*, this is? Close, not two *x* though. So, if this is an even number like eighteen, this is supposed to be the next even number, what would that next even number be?
>
> Carol: Oh, eighteen plus two.
>
> Mr. Anderson: Yeah, plus two, so it would be *x* plus two.
>
> Carol: Quantity squared?
>
> Mr. Anderson: Mhm.
>
> Carol: I was close. (Observation, June 28)

Mr. Anderson provided Carol with numerical examples to try to help her make sense of the situation and construct a correct equation. While both Mr. Anderson and Carol said she was "close," her initial mistake was to use the same variable for two different unknowns, which was not close but a conceptual error indicating she lacked understanding of the role of unknowns in algebra. This was especially surprising since she had three years of experience teaching high school algebra.

In summary, the most salient roles played by Mr. Anderson were to explain thoroughly and be supportive; his goal was to provide students with the tools they wanted to learn. Mr. Anderson strived to maintain a relaxed and comfortable environment, demonstrating support for students and their preferred ways of learning in spite of the short term and the number of topics they covered. Students clearly indicated they wanted to see examples, hear Mr. Anderson's explanations and have opportunities to ask questions, and Mr. Anderson provided those. Rasmussen et al. (2003) studied two college differential equations classes, one in which students were expected to explain their answers to the other students in the class and listen to each other; the teacher specifically discussed norms and roles at the start of the term and then fostered these roles. In the other class, the teacher played a role similar to Mr. Anderson's by affirming or refuting answers and being the only explainer in the class.

Negotiating Students' Roles

Students' roles in relation to their expectations. Students' responses to the questionnaire indicated they expected limited roles in mathematics classrooms. When asked if they liked mathematics teachers to involve the class in discussions, eleven of the thirteen students indicated class discussions helped them better understand (see Table 24). However, when asked if they offered input during class discussions and what types of input they usually offered, only seven students responded “yes,” but added that the type of input they offer is to ask questions (see Table 22). Four students responded they would not usually contribute during whole-class discussions while only Janet and Thomas indicated they would contribute anything except questions. Janet wrote she would contribute if she could "show that I know what is going on" (Janet Questionnaire, June 27), but Thomas did not describe the types of input he would be willing to provide (Thomas Questionnaire, June 27). So, while most students liked class discussions, fewer were willing to participate, and those who were willing perceived their roles as providing correct answers or asking questions, rather than offering ideas to be explored by their peers.

Some students wrote they preferred not to join in whole-class discussions and rarely participated. Sam and Kathy responded that they would not usually offer input, and rarely contributed in class unless they were called on. Sam responded that he would ask after class if he had a question (Sam Questionnaire, June 27) and the only time I observed him ask a question for the whole class to hear was during a class period when Mr. Anderson told the class to take five minutes and explore the chapter to come up with questions (Fieldnotes, July 19). In response to whether she would participate in whole-class discussions, Kathy replied, “No, I have a hard time giving input in a math class” [underlining in her response] (Kathy Questionnaire, June 27). For the most part, she restricted her interactions in class to working with Anthony during opportunities for group work and asking Mr. Anderson to do homework problems. However, twice late in the quarter she offered short answers to closed questions (Fieldnotes, July 27; Fieldnotes, August 2). While Kathy originally responded emphatically that she would not contribute during a mathematics class, she supplied correct answers when it appeared no other student was going to provide them. She may have been willing to contribute because it did not require much risk and she had become comfortable in this class.

Impact of relationships on students' roles. Anthony also responded he would not usually offer input in class, preferring class with lecture followed by whole-class discussion (Anthony Questionnaire, June 27) but did become involved in the discussions on several occasions. I coded his participation on six different days after July 11, each time with several contributions (e.g. Appendix K, Lines 26-32). So, while some students indicated on their questionnaires they preferred not to participate, they contributed when opportunities were provided.

Some students who wrote they would participate in class discussions by asking questions (Student Questionnaires, June 27) remained quiet during whole-class discussions. I did not observe Greg or Sheila participate in whole class discussions other than to ask for homework solutions, and Brian volunteered responses to questions on only one day I observed (Fieldnotes, July 19). Requesting homework solutions was not coded as students' questions or as participation. Each of these students worked alone when given opportunities to work with their peers. Greg sat at the back left of the room, away from other students, and I never observed him or Brian talk to other students. Sheila sat near the back middle of the room but was absent more than two-thirds of the classes and did not talk to other students when I observed. Although these three students responded they would be willing to participate, they did not. It may be that they did not have any questions, they waited for other students to ask similar questions, or they did not feel comfortable enough to ask.

However, Mr. Anderson called on the quiet students several times throughout the quarter. An example of a typical episode:

Mr. Anderson: Okay, how about uh, *g* of negative seven? All right, you guys try to find *g* of negative seven and then we'll put the answer on the board [pause 20 seconds]. All right, Greg, what did you get?

Greg: Twelve.

Mr. Anderson: Twelve? Anybody else get twelve? Good, all right. Questions on that? (Observation, July 7)

When Mr. Anderson called on students who had not volunteered, it was to answer closed questions after everyone had been given time to think. This practice set an expectation for all students to work on problems presented during class time. Mr. Anderson may have hoped by calling on these students, they might feel more willing to speak up without being called on in the future. However, Greg did not volunteer answers during my observations.

Two students' answers to whether they would offer input in whole-class discussions and what types indicated that how the students felt about themselves in the class determined whether they would contribute. Carol replied, "Sometimes, depending on confidence, questions" (Carol Questionnaire, June 27), and Kevin responded, "Usually not because I am a slow learner" (Kevin Questionnaire, June 27). Both responses implied the students would not contribute during class because they were concerned about others' perceptions of them. However both students regularly contributed during class, providing evidence they felt safe in this classroom environment (e.g. Observation, July 27). Carol often worked with Sarah, and Kevin usually talked with Thomas during opportunities for group work. Perhaps being comfortable enough to talk to those nearest them helped students feel comfortable enough to speak up during whole-class discussions.

Some students contributed often during whole-class discussion: Daniel, Carol, Sarah, Thomas, and Kenny, and to a lesser extent, Kevin, Anthony, and Janet. Other than Anthony, these same students sat near each other and engaged in conversation regularly during opportunities for group work. Anthony worked with Kathy during group work. Students who did not speak to other students during opportunities for group work also did not contribute during whole-class discussion. Inversely, if students did speak to their peers during opportunities for group work, they also contributed at least occasionally during whole-class discussion. The criterion of talking to others nearby was a better indicator of students' willingness to contribute during whole-class discussions than their responses to the questionnaires. That is, students' willingness to participate in whole-class discussions may rely on the extent to which a community with their peers is formed.

The nature of students' participation. While many were willing to participate, their conceptions of the ways they should participate were limited. While every student wrote "yes" or "sometimes" to the question: *Does listening to other students' questions or explanations help you learn?* (see Table 23) only Kevin responded that other students' ideas or solutions might be valuable; nine answers to this were "yes" but referred to other students' questions and the teacher's answers. "Yes, to questions; I don't like it when students show how they do the problems" (Greg Questionnaire, June 27). Daniel added that other students' inputs could confuse him (Daniel Questionnaire, June 27). Greg and Daniel's responses indicated they had been in classes where other students shared solutions or explained, but did not like it. This shows that while most students wanted whole class discussion, and even those who did not like to take part in class discussions wanted to listen to it, students wanted to hear explanations from the teacher.

Correspondingly, students did not offer their ideas for other students during whole class discussion. Students' questions and answers during whole-class discussions were intended for the teacher, not to contribute to other students' learning. Thomas, Janet, and often Kenny, sometimes spoke so quietly only Mr. Anderson and those nearest them could hear. It is likely that students who valued the teacher's explanations above other students' explanations may have believed other students would not want to hear their explanations, and Mr. Anderson's practice of repeating correct answers and explaining enabled students to maintain their roles.

Many of the questionnaire responses also indicated some students expected to have a passive role in class: the teacher should do examples, explain in detail and write clearly on the board. Students' answers to what they could do during class that helped them learn included "take notes" for six students (see Table 25). Others referred to efforts outside of class such as "homework" and "examples, study groups" (Student Questionnaires, June 27). Mr. Anderson's approach to teaching was helpful to those who valued taking notes since his work on the board was organized and he did not erase often. His slow pace gave ample opportunity for students to be able to write thorough notes.

A couple of responses indicated that students wanted a more active role during class, "Attempt to work problems before teacher completes it" (Kenny Questionnaire, June 27) and "hands on examples" (Brian Questionnaire, June 27). Both responses indicated students wanted to be able to work problems in class rather than just watch the teacher work problems, an indication that they were absolute knowers with a mastery orientation (Baxter Magolda, 1992). Observation evidence indicates Kenny strived to meet this goal (see Appendix L, Lines 20 and 22). In this example, he worked ahead of the class during discussion and either gave the answer to check its correctness with Mr. Anderson or to show what he knew. The answers were too brief to be helpful to other students and were expanded on by Mr. Anderson for the rest of the class. Thus, Kenny's perception of his role in class did not include contributing to others' learning, but rather making sure he could complete the procedures correctly.

Sarah, Kathy, and Carol all responded, "practice" to what they could do during class to help them learn (Student Questionnaires, June 27). Throughout the quarter Mr. Anderson offered many opportunities for students to practice with his feedback and with others around them. I have referred to these activities as *seatwork* and discuss them more in-depth in a later section.

Students' ways of knowing. Students' ways of knowing could be inferred from their responses to their questionnaires and their behaviors in class (see Table 26). Carol's questionnaire responses and behaviors provided evidence of absolute knowing; her comments and questions indicated she thought there was a specific procedure for any mathematics problem or question:

> Carol [to Mr. Anderson]: When you ask questions, like, I don't know if there'll be anything like this on the test, but range for the window, how exact are you?
>
> Mr. Anderson: Well I won't ask, give me a range for your window.
>
> Carol: Okay, it seems silly.
>
> Mr. Anderson: What I'll ask for is give me a nice graph with all the features…
>
> Carol: Number eleven.
>
> Mr. Anderson: Number eleven on one point nine? He reads it. ["Find an appropriate viewing rectangle for $y = \sqrt{256 - x^2}$ " (Stewart et al., 2002, p. 112).]
>
> Carol: Yeah.
>
> Mr. Anderson: So, is your question the fourth-root part?
>
> Carol: I just didn't know what to do with it.
>
> Mr. Anderson: Okay, to type it in or to get it to graph? Did you get it typed in your calculator?…[discussion continued until they finished graphing it.]

Carol: So we can just do that and draw the graph? I guess, I was trying to figure it out on paper. (Observation, July 5)

It was apparent Carol had graphed it correctly, but she seemed to think there was a way to determine an exact viewing rectangle on paper. Mr. Anderson used –25 and 25 for *x*-min and *x*-max of his viewing window while the solution in the back of the book showed a window with *x*-values from –20 to 20, and *y*-values from –1 to 5. Earlier in the same class period, Carol and Sarah discussed a homework problem consisting of a graph students completed assuming it was symmetric about the *x*-axis. Carol said to Sarah, "I could figure this out, but I couldn't do it mathematically" (Fieldnotes, July 5). Carol's comments indicated she was an absolute knower since absolute knowers believe that the nature of knowledge is certain and that students acquire knowledge from some authority, such as the teacher or textbook (Baxter Magolda, 1992).

Absolute knowers also have difficulty discussing concepts and are unable to apply mathematics out of given contexts (Brew, 2001). In the following episode Carol asked Mr. Anderson if he would do a certain type of problem, which implied he must show them how to do each type of homework problem.

Carol: Are you going to go over today, how to do problems like twenty-nine through thirty-eight? ["Find a polynomial with integer coefficients that satisfies the given conditions" (Stewart et al., 2002, p. 306). Each exercise included conditions of a degree and two or three zeros.]

Mr. Anderson: Uh, we can. Which one are you interested in?

Carol: Twenty-nine [degree two, and zeros $1+i$ and $1-i$.]

Mr. Anderson: Twenty-nine. So we want degree two with zeros one plus i and one minus i. [Repeats as he write it on the board.] Okay, so if this is a zero, what's a factor? [8-second pause.]

Carol: x minus one plus i?

Mr. Anderson: Right.

Carol: x minus one minus i. (Observation, July 27)

Mr. Anderson addressed Carol's question by asking a question to help her make a connection, although she did not answer quickly. This problem was not very different from a problem discussed in class two days before of finding all polynomials with 7 and 3 as zeros (Fieldnotes, July 25). In addition, just before Carol asked her question, the class discussed a problem finding the zeros of a fifth-degree polynomial, some of which were non-real, and writing the polynomial in factored form (Appendix K). However, the way Carol asked the question implied that she did not expect to be able to do these exercises unless Mr. Anderson explicitly discussed how to do them.

Carol also asked questions she could check herself. For example, she wanted to check her answer to finding the equation of a circle given endpoints of a diameter. Mr. Anderson responded by explaining his process so Carol could compare hers, consistent with the roles of this class, rather than ask her to explain what she did and why (Observation, July 5). Carol could have checked that her equation satisfied the criteria by substituting in the ordered pairs or by using her graphing calculator, but she did not do this and Mr. Anderson did not suggest it. Consequently, Carol's beliefs about an instructor's role and Mr. Anderson's acceptance of that role interfered with her development of agency to make sense of the mathematics.

Sarah also strongly evidenced absolute knowing since her questions indicated she believed learning mathematics consisted of learning rules (Brew, 2001). Several of Sarah's questions indicated she was seeking a rule even when she could understand. In response to students' request to give them a "tough one" to complete the square to put a quadratic function in the form $y = a(x-h)^2 + k$, Mr. Anderson provided the example: $y = -4x^2 - 2x + 7$ (Fieldnotes, July 13). After rewriting it in the form $y = -4\left(x^2 + \frac{1}{2}x\right) + 7$, Mr. Anderson explained by adding $\frac{1}{16}$ inside the parentheses, they subtracted $\frac{1}{4}$ from the right side since it is multiplied by -4, thus they needed to add $\frac{1}{4}$ outside the parentheses to keep the same value. However, Sarah asked, "If it is positive, would we subtract on the outside?" (Fieldnotes, July 13). Rather than understanding and using reasoning each time to determine whether she would need to add or subtract, Sarah preferred to have a rule.

Sometimes Sarah's use of rules conflicted with new knowledge:

> And I think it was over, over today's stuff, when we were doing the vertex, how it's the negative, [pointing to $y = 2(x-3)^2 - 4$ in her notes] well the three and the negative four. Well, I just assumed from circles, doing *h*, *k*, that it would be the opposite, so [the vertex would be at] three, four. So that's what I was thinking, and so when he first wrote that down I was like, well that doesn't make sense, but then after we wrote down what we needed to do to the parabola [shift right three and down four] then I was like, of course, if you're starting at zero, zero as your starting point and you move, you know, to the right three and then down four, and if I graph that, then that's where the vertex would be, so I had to write this down, yeah, cuz when he first said that I was like, well wait a second

that's kind of opposite from what we were doing with circles, cuz circles you do the opposite signs as the center. (Sarah Interview, July 13)

Sarah initially compared the form of the quadratic function to a rule for locating the center of a circle from its equation, apparently connecting the use of the letters *h* and *k*. However, when the rule did not work and Mr. Anderson justified the location for the vertex using transformations, she said it made sense. That is, Sarah said it “made sense” when she knew which rule to choose based on Mr. Anderson’s explanation; she did not pursue her initial idea by questioning why the rule for equations of circles differed from the rule for transformations. Thus, “making sense” takes on different meanings for different ways of knowing. In Sarah’s case, it means knowing which rule to choose based on a single explanation from an authority whereas to contextual knowers, it means resolving perturbations in their own ideas while taking into consideration others’ ideas.

Challenging students’ roles. While, for the most part, students’ roles were to ask questions and listen to Mr. Anderson’s explanations, early in the quarter Kenny answered Sarah’s question during a whole-class discussion:

Sarah [to Mr. Anderson]: Um, I do not remember, but how can you tell, just by looking at the equation, whether it opens up or down?

Mr. Anderson: Oh,

Kenny [has eagerly raised his hand]: The sign of the leading coefficient.

Mr. Anderson: Yeah, exactly, if you're talking about squared. (Observation, July 7)

Mr. Anderson continued his explanation to include end behavior of polynomials in general. Kenny’s response was consistent with the idea that students speak up to “show what they know.” His answer was not complete enough for Sarah to use as a rule since he did not say how the sign affected the direction of the parabola. Mr. Anderson affirmed Kenny's answer and, although Sarah's question was specifically about parabolas, discussed end behavior of polynomials in general. While this would have been a good opportunity to have all students explore why the sign of the leading coefficient affects the end behavior of the graphs of quadratic functions, Mr. Anderson's response maintained his role as the primary explainer of mathematics in the classroom and students' roles as accepting his authority.

However, later in the same lesson when Mr. Anderson drew a graph on the board and asked the class to evaluate a function for a specific value, Mr. Anderson pressed Kenny to explain.

> Mr. Anderson: All right now, from this graph, let's see if we can find some things: like what is *f* of negative three? I don't have a function rule to plug it into; I just have a graph.
>
> Kenny: Three.
>
> Mr. Anderson: It's three! Excellent. How'd you get that Kenny?
>
> Kenny: *f* of negative three is three.
>
> Mr. Anderson: That's true *f* of negative three is three; how'd you get that, go ahead
>
> Kenny: Because, you said before that that's your *y*-value. [OC: Kenny seemed to be uncomfortable with the pressing.]
>
> Mr. Anderson: Yes, right this is the *x*-value; this is the *y*-value, so how'd you get that from this graph?
>
> Kenny: It's a point.
>
> Mr. Anderson: Okay, yeah, there's this point over here, negative three, three. That point's on the graph; that means that has to be the function value. [Observation, July 7]

It appeared Kenny did not want to give an answer beyond what he had volunteered. He continued giving answers that needed to be expanded to be useful to other students. Although it was only the seventh class day of the quarter, it was not the role of students in this class to explain for other students and Kenny resisted an attempt by Mr. Anderson to initiate this expanded role.

How students believed they learn. Students' questionnaire responses showed they believed learning mathematics consisted of memorizing procedures demonstrated in class and assigned for homework. According to their responses, eleven of the thirteen students believed memorizing steps and formulas was important in mathematics (see Table 27). Of the two students who wrote "no" to whether or not they found memorizing steps and formulas important, Daniel wrote, "no, formulas can always be looked up. Steps are good to memorize" (Daniel Questionnaire, June 27) while Carol responded, "No, notes help more" (Carol Questionnaire, June 27). In response to what they do to understand the ideas, where the formulas come from, and why they take the steps they do, two students replied they did not need to understand while eight other students responded they needed repetition and explanations (see Table 28). As their responses illustrate, the students in this class had a predominantly traditional view of learning mathematics (Smith, 1996).

While Daniel indicated he relied on memorization to learn mathematics, there is evidence his memory was unreliable. In one instance, he said he would not be able to remember transformations of functions without looking at his notes (Fieldnotes, July 13). Also, when he needed to find the area of a triangle, he asked Mr. Anderson if he should use "*a*-squared plus *b*-squared equals *c*-squared" (Observation, July 5). He recalled a formula he associated with triangles, the Pythagorean Theorem, rather than with particular characteristics of triangles, demonstrating pseudo-conceptual behavior (Vinner, 1997).

Mr. Anderson also emphasized knowing procedures; on the first day of class he told them if they could do all the homework problems without referencing the book or notes, they would do well on the exams:

> When you do the homework, there's two stages on the homework. The first time you're going through doing some of the problems and you're referencing your notes, the book, other people, tutors, whatever, but you gotta get to the level of doing the problems without referencing any of those things. So if you just do the first level without practicing the second level, you'll have trouble with the exams, okay. If you get to that second level, when you can do those problems without referencing anything, then you're good to go, that's kinda where you've gotta aim to be. (Observation, June 27)

This warning could be interpreted as an encouragement to students to memorize each procedure. However, Mr. Anderson usually offered reasons for procedures he introduced indicating he thought understanding was important.

Summary of students' roles. Students entered this class with experiences and preferences for their roles. Some believed their legitimate participation in this class should be limited to answering their instructor's questions, asking questions, and working practice problems they had been shown how to do. Students did not expect to contribute to others' learning during whole-class discussion or that their ideas should be the focus of discussion. They did not value listening to their peers' ideas during whole class discussions since it was Mr. Anderson's role to explain the mathematics. For the most part, these ideas were not challenged. However, students did not remain true to their original preferences: if students interacted with their peers during seatwork they were more likely to contribute during whole-class discussion, while if they did not interact with others, they rarely contributed during whole-class discussion, regardless of what they answered on the questionnaires.

The students in this class demonstrated absolute knowing on their questionnaires and through their behavior in class. Absolute knowers can be further categorized as *receivers* or *masters* (Baxter Magolda, 1992; Brew, 2001). In general, the students who rarely spoke up in class were receivers while the students who contributed often during whole-class discussion were masters. In either case, as absolute knowers, they believed that mathematical knowledge was certain, the instructor had the knowledge and their job was to obtain it. This implied Mr. Anderson's role was to communicate his knowledge clearly and ensure that students understood it.

Baxter Magolda (1992) suggested that teachers need to teach responsively to students' ways of knowing, but after building connections to students, teachers must balance this confirmation with contradiction of students' ways of knowing. For example, to confirm students' absolute ways of knowing, Mr. Anderson demonstrated helpfulness and support, roles absolute knowers expect of good instructors. However, to contradict students' ways of knowing, Mr. Anderson could have included more opportunities for students' ideas to be central to class discussion, encouraged students to listen to and respond to each other, and restrained his role as mathematical and intermediate authority. Such balance, according to Baxter Magolda, helps students develop more complex ways of knowing.

Negotiating Social Norms of Whole Class Discussions

Class time was used in two ways: whole-class discussion and seatwork. In this section I describe classroom social norms during whole-class discussions and in the next section, social norms during seatwork. In general, very little lecturing without input from students occurred in this class because the social norms and roles were that students should speak up whenever they had something to share or ask, and Mr. Anderson provided many opportunities with his questions, pauses, and wait-time.

Mr. Anderson's value of student participation. Mr. Anderson explained that his goal in all his classes was to maintain a relaxed and friendly atmosphere and the sense of a leisurely pace (Mr. Anderson Interview, August 16). He learned students' names during the first week of class and used them throughout the quarter. He usually started class by answering homework questions thoroughly and slowly so students could work the problems with him. In addition, throughout the lesson he paused and encouraged questions. Mr. Anderson's intent was to cultivate an interactive classroom.

> I like to have a very open environment where I hear a lot, you know, students are often so afraid to ask that question, but no, no, you have a question, you just gotta get it out there, because most likely half the rest of the class has the same question they're just not asking as well. I try to be really open to questions and, and encourage interaction in my class. You know, we say we lecture, but, I don't know, it's more an interactive thing with me. I put it on the board, here. So, I like to do that where, like even just small problems, say, okay, what do we do next? Or after I do the first one, I say, okay you guys help me with the next one. So, it's an interaction with the class. (Mr. Anderson Interview, July 5)

Mr. Anderson knew most students wanted to leave class knowing how to do the homework and wanted opportunities to ask questions and receive clear explanations. He also believed that if one person had a question, others also wanted to hear the answer. Students' responses to their questionnaires corroborated this idea (see Table 23) since nine of the thirteen students replied they liked hearing other students' questions and five specifically mentioned others asked questions they were thinking. Because he was responsive to students' preferences, he continued to encourage questions and provide opportunities for students to participate as they chose.

However, Mr. Anderson also explained he was not always able to get students to interact in class:

> Well, sometimes, yeah, for whatever reasons, you get a class that they do not want to interact like I would like to have my classes interact and then, then it's a struggle with, well, what do you do then? And some classes I just, like, three, four weeks into the quarter, if they're just going to be that way, then I just have to, I guess what I'm taking that as, maybe they do not want to learn through that interaction, they just want to learn through observing, then I'll change my style a little bit and not make it so interactive. Because, well it's for one, pretty tiring trying to pull answers from somebody who for four weeks has not opened their mouth with any words. It's fine, like, you know, I mean, the interaction for me is done to help them and if they don't choose to take that. I mean, that's another thing you could say clear back on the philosophy is, it's my role as a teacher to give the tools to succeed, yet if the student doesn't want to use all those tools, that's their choice at this level. (Mr. Anderson Interview, July 5)

While Mr. Anderson valued interaction as a style of teaching possibly helpful to his students, if students did not want to interact, he did not believe it was his role to force them. He tried to teach in ways students expected and offered them what they needed, but it was up to the students to choose to use the opportunities in ways they believed would benefit their learning.

I also asked him to describe a particularly successful class and explain what made it successful:

> I think they did a lot of things, they put the effort in outside of class, and maybe that came from enjoying what we were doing in class, not necessarily just the work, but the interactions maybe. So, if they enjoy that, then they're willing to go home and do the work they need to do on it. Yeah, last year I had this group that followed me through precalc one and precalc two, and then a smaller core of that same group went calc one, calc two, but it was a very, very nice core group, that by the end, you know, we were taking three or four days to do what used to take us five days because they would just get into it. We could really cover the material. (Mr. Anderson Interview, July 5)

When he had a group of students who became familiar with him, each other, and the social norms of the class, they interacted in ways that accelerated learning. It may be that these students had developed more complex ways of knowing since they were more advanced in school, or that the relationships they formed over time enabled better communication.

Initiating participation. Mr. Anderson initiated a social norm for whole-group interaction on the first day of class by frequently asking questions and waiting until he heard an answer to proceed. At one point while solving an equation he paused and waited at the board quietly and then said, “I hope somebody's going to help me out with the rest of this.” A student responded with an answer, although it appeared students were quiet because they were each solving the equation by themselves (Fieldnotes, June 27). After that, there was a social norm that students tell him the next step without his asking when he was at the board and they were working procedures. Often, the pace of his speaking slowed or he paused, indicating to students that it was a good time to ask a question or give an answer. Several students participated quickly in this social norm, indicating it was a social norm familiar to them.

Students soon used any pause in the discussion to ask questions. On the first day of class during a long pause, Daniel asked, "Can you run through that again?" Mr. Anderson worked a related but simpler example, during which Daniel said, "Okay, I got it" (Observation, June 27). But Mr. Anderson finished the example, initiating a social norm that he would respond to students' requests for explanations, but once he started explaining, the explanation was offered to all students and would not be stopped because the student who initiated the question was satisfied. This teacher behavior is consistent with what students expected of their teacher, since most students indicated on their questionnaires that they learned from listening to other students' questions and the teacher's answers (see Table 23).

During whole-class discussion, Mr. Anderson usually responded to correct answers by repeating them as he wrote them, often nodding affirmatively, or saying "yeah" (e.g. Observation, July 11). Since many students' answers were not loud enough for the whole class to hear, repeating them as he wrote on the board validated them, made them part of the whole-class discussion, and connected the symbols he wrote to the language he used. Early in the quarter, Mr. Anderson responded to wrong answers with "close," (e.g. Observation, June 27), which appeared to encourage students to continue answering, while later some wrong answers were rejected with "no" or refuted with reasons. By determining whether answers were right or wrong during whole class discussion, Mr. Anderson accepted a role as mathematical authority.

Students' initiation of social norms. On the fourth day of class Daniel initiated a norm of asking for specific types of examples by asking to see a problem that required students to find the equation of a circle tangent to the *x*-axis (Fieldnotes, June 30). From Daniel's response to his questions, Mr. Anderson realized the idea of a circle tangent to the *x*-axis contributed to Daniel's confusion, so he drew several circles both tangent and not tangent to the *x*-axis to help students understand the idea and then finished the problem with input from students. The accommodating way he responded to Daniel's request encouraged students to ask for specific examples throughout the term and was consistent with his roles as explainer and supporter.

Daniel also initiated a norm of asking questions to verify his thinking early in the quarter. On the fifth day of class, a whole-class discussion focused on finding the equation of a line through two points. Taking students' suggestions, Mr. Anderson found the equation using the point-slope form of the equation. During the pause that followed, Daniel asked, "Can't you also plug the negative seven and the eight into the *y* equals *m x* plus *b*, then solve for *b*?" Mr. Anderson replied by following this suggestion to demonstrate how the substitutions could also be used to find an equation. After asking if there were questions, he pointed to Kevin, who had not raised his hand but had a confused look on his face, and said, "Thinking about something?" to which Kevin responded, "We can just plug it into any of those equations?" Mr. Anderson then compared finding equations using each of the three different forms of linear equations, focusing on how many unknowns would be left and what type of information they had (Observation, July 5). Other students also asked questions introducing new ideas (e.g. Observation, June 30; Observation, July 7) and Mr. Anderson responded with clear explanations. Thus, while students introduced ideas, they were usually in the form of questions and the social norm was that Mr. Anderson explained the mathematics.

While most student input was in the form of questions, Daniel, and later others such as Carol and Sarah, also spoke out during whole-class discussions with comments or insights. In the following episode, discussion focused on completing the square to rewrite a quadratic function in vertex form: $f(x) = 2(x^2 + 4x + 4) - 5 - 8$

> Mr. Anderson: Okay, the four multiplied by the two is eight. So what we really did is we just added eight to this side, and then we subtracted eight. So what we've done is we've added eight and subtracted eight.
>
> Daniel: So we've done nothing.
>
> Mr. Anderson: We've done nothing to this side, right. Our goal is to not to change anything. We added eight so we subtracted eight, so we haven't changed it. (Observation, July 13).

The social norms of this class allowed Daniel to check his thinking and clarify Mr. Anderson's explanation. Mr. Anderson affirmed Daniel's comment and repeated it to the rest of the class. While students were encouraged to contribute their thoughts and questions during whole-class discussion, students' questions and comments were of a limited nature. This idea will be explored more in-depth in a later section.

Students who remained quiet. When discussing student interaction with Mr. Anderson, I asked about Janet, mentioning that she was very quiet.

> Yeah, but she is on the ball, definitely pays attention to what we're doing. I think she likes to think about it. So she likes to take her time and really digest it and say okay, what do I need to do with it to get it to work, to understand, rather than bouncing ideas. So it's not necessarily a problem if they don't interact, you know, it's just a different way." (Mr. Anderson Interview, July 5)

So, while Mr. Anderson valued interaction, he did not believe students who did not interact during whole-class discussion were not thinking and he respected their preference to stay quiet. In general, we discussed students who did not speak up much during class discussions:

> Yeah, or they might just have it inside, or sometimes if I can, if I hear somebody, like under their breath, I'll like pull it out, like, okay, what was that? Say it louder so everybody can hear it, you know, if you can, but sometimes it's hard when they're spread out like that. (Mr. Anderson Interview, July 5)

I observed an example of this on the second day of the quarter: Mr. Anderson asked if someone had simplified the expression on the board and he thought Kathy indicated yes by nodding. However, when he called on her she clearly did not want to answer (Observation, June 28). Mr. Anderson valued whole-class interaction, although he understood its limitations; some students may not want to interact, and being spread out may discourage some students from participating. Since students chose where they sat, some may have chosen to sit far away from the others to avoid participating or interacting with others.

Nature of whole class discussion. During whole-class discussions, Mr. Anderson worked slowly with long pauses so students could work problems along with him. An episode on July 27 (see Appendix K) illustrates the amount of time Mr. Anderson allowed for working homework problems. During this episode, Mr. Anderson did most of the talking and explaining. However, students were expected to work the problem along with him, to try to solve the problem by graphing the polynomial on their graphing calculators and try to understand as the problem was discussed. While they worked along with him, asking questions and offering answers, he did the math and affirmed answers. For example, when Sarah said "three," Mr. Anderson followed with "Close, three," and waited for someone else to finish. Also, while he gave Anthony a chance to give his reason for knowing the exponent had to be odd, Mr. Anderson still clarified his answer for the rest of the class. In addition, Mr. Anderson mediated rather than facilitated discussion since each student statement or question was directed at him and he either explained, restated, rejected, or validated it, and he was the only one to speak to the students.

Opportunities provided by whole-class discussion. The social norms of whole-class discussion provided students with opportunities to ask questions or make statements when they wanted further explanations. For example, on several occasions students asked Mr. Anderson to re-explain. On July 11, twice students spoke up to ask for either a review of a procedure, or for more explanation on an idea that was just introduced: Daniel asked for a review of finding the range, and then, while Mr. Anderson was introducing direct variation, Kevin spoke up to ask, "*k* is what again?" (Fieldnotes, July 11). To Kevin's question about the constant of variation, Mr. Anderson produced an example of the circumference of a circle, a formula probably familiar to Kevin. Mr. Anderson's responses generally encouraged these questions since he always answered them fully.

As the quarter progressed, Mr. Anderson adjusted instruction whenever students indicated they were confused. The following episode happened near the end of the quarter after Mr. Anderson introduced mathematical induction by providing domino and ladder analogies, then presented an example: Prove $P(n): 2+4+6+...+2n = n(n+1)$ is true for all natural numbers *n*.

Mr. Anderson: We want to prove that this is true for all natural numbers *n*. …

We're going to assume that it's true for *n* equal *k*. Kenny, you have a question? [From fieldnotes: both Kenny and Carol are visibly looking confused; Kenny put his pencil down and leaned back and folded his arms as if he could not go on.]

Kenny: Uh, your *P* sub two,

Mr. Anderson: *P* of, you want to do *P* of two? Okay, before we do this, let's do *P* two.

Carol: Yeah, this is silly.

Kenny: Wouldn't your left-hand side be four?

Mr. Anderson: No, our left-hand side would be six [Moves to the other board to write]. Okay, if you're looking at, this is scratch work,

Kenny: Six, I think that would be *P*-three, wouldn't it?

Mr. Anderson: No, no, okay, let's look at *P*-two, our second statement. So we're going to add on the left, two plus four plus, we're going to end when *n* is two, so we're going to do, our left-hand side,

S: You have to add it up.

Mr. Anderson: Is two plus four, is six.

Kenny: Okay.

Mr. Anderson: My right hand side is two times two plus one, which is also six.

Kenny: Okay [he picks his pencil up and leans forward in his chair]. And so *P*-three would be twelve?

Mr. Anderson: Let's do *P*-three. Our left-hand side would be two plus four plus six. Why am I stopping at six?

S: Because that's your third term.

Mr. Anderson: Because that's our third term in our sequence. We're supposed to do plus four; we're supposed to stop when we get to two times three. (Observation, August 15)

Kenny and Carol clearly indicated their confusion after Mr. Anderson assumed the induction hypothesis. Mr. Anderson watched students' reactions to try to determine when they were confused and was willing to explain and backtrack until students indicated they understood. In response to their frustration, Mr. Anderson stopped writing the proof and wrote each of the two statements on another part of the board, discussing their meaning. This social norm of asking questions whenever they were confused may have helped students who were trying to understand throughout each class period. Sarah discussed how this social norm helped her:

> I make sure that I understand it as I go along because I've learned from trying to take precal on my own before, that if don't ask questions and I don't understand before I go home and try to do it, that I'll just be lost… and I think class size has a lot to do with that, and the instructor, feeling that I can approach him or ask him if he's available. (Sarah Interview, July 13)

Sarah suggested the nature of whole-class discussion, aided by a small class size and an approachable instructor helped her learn in a way she could be successful. She admitted she had few resources if she attempted to do problems at home before she understood. This was consistent with her response to how she best learns math: "practice and a good instructor" (Sarah Questionnaire, June 27) and consistent with absolute knowing since absolute knowers attribute their learning to their instructors (Baxter Magolda, 1992).

Sarah also added why she felt comfortable enough to participate in this class:

> I think the small class size is also important, because I don't think I would feel as comfortable if we had a large class, asking as many questions as I would, to try to understand something, …I just the think the way he introduces the topics in the class, I think he explains them very well, and he does multiple examples and that kind of helps

> me learn. I mean, you can show me something, but I need multiple examples of how we get there before I feel comfortable doing so, right?… In fact, [Carol] today was like, 'I know this is stupid but,' he's like, 'no, no there's no stupid question.' And I think students are all, just by being a teacher, they also like, they don't want to talk or ask questions because they think it's a stupid question, … You know what, if I'm thinking it, then obviously maybe somebody else is thinking it, so I might as well ask it. When I haven't asked questions that I've been thinking, somebody else will ask it.…. Another thing I like about him is he makes sure we all understand it before we move on. If somebody doesn't understand it, he's willing to show them so we can all move along as a class. (Sarah Interview, July 13)

Several factors contributed to Sarah's willingness to ask questions: her maturity, her identity as a teacher, the small class size, and Mr. Anderson's openness to questions and thorough explanations. She also appreciated that Mr. Anderson wanted all students to understand and was willing to continue explaining or come up with examples until they indicated understanding.

Summary of social norms of whole-class discussion. The social norms of whole-class discussion supported discussion between the teacher and students rather than a discussion between all members of the class, students expected to be able to ask questions of Mr. Anderson and have them answered until they indicated they understood, and only those students who wanted to, participated. Also, students were more likely to contribute answers if they thought they had a correct answer, if they wanted to show what they knew, or if they had a question. Concomitantly, Mr. Anderson did the mathematics, approving or rejecting students' answers. Because these social norms did not contradict their beliefs that the teacher had knowledge and could give his knowledge to them, their absolute knowing was not contradicted.

Social Norms of Seatwork

The other major class activity is referred to as seatwork since it was both individual and group work depending on how each student participated. Seatwork occurred when Mr. Anderson assigned one or more problems and gave students time to work on them. He assigned seatwork on 14 of the 19 days I observed, and the times allotted for seatwork each day ranged from a few minutes to around fifty minutes. During seatwork, all but three students, Greg, Brian, and Sheila, regularly discussed mathematics with students near them.

Initiating seatwork. On the first day he assigned seatwork, Mr. Anderson told students they were free to work together:

> All right, rather than me sitting up here and doing a bunch of these, I think on these types of problems what I want to do is just wander around and see where you're at. So, we're going to do multiples of twelve in section one-six. If you don't have your book, that's okay, you can just work together, just scoot your desk over, and look at somebody else's book. So, we're going to do multiples of twelve in section one-six, [writing on board] twelve, twenty-four, thirty-six, forty-eight, sixty, up to seventy-two [for assigned problems, see Appendix M]. So just use scratch paper, I'm not necessarily going to collect them. So if you get stuck raise your hand, I'll come around. You can work together. (Observation, June 28)

Mr. Anderson had not done any examples for this section on modeling with equations. He assigned this seatwork so he could assess what students knew about solving problems in this section; opportunities for students to work together were secondary. By telling them he would help if they raised their hands, he demonstrated his helpfulness, important for absolute knowers (Baxter Magolda, 1992) but also retained authority.

During this first opportunity for seatwork, Kathy and Anthony immediately moved their desks together, as did Carol and Sarah (Fieldnotes, June 28). Both pairs talked in low voices. During the next forty minutes, Mr. Anderson walked around the room looking at students' work and responding to their questions. Also, as the time progressed, other pairs of students started working together, including Thomas and Kevin. Thomas also turned and talked to Sarah, Carol, and Kenny about the mathematics. All but four students talked to other students during this time (Fieldnotes, June 28). During this particular episode, Thomas asked Mr. Anderson a question:

> Mr. Anderson: First thing, I see sum, you did product. [Thomas erases something on his paper]. So, sum is going to mean add. … Now did we square the second number? There you go, now we have the sum of the squares. [Mr. Anderson continues walking around the room.]
>
> Carol [to Mr. Anderson]: Does that look right?
>
> Mr. Anderson: Yeah, that looks great, keep going.
>
> Carol: So now I just [inaudible]?
>
> Mr. Anderson: Yeah, now you can either factor it, or use the quadratic formula, or complete the square. [He continued to walk around]…Oh, oh not just two. Yeah, *x* plus two…Square, not square root. (Observation, June 28)

Carol's question exemplified the typical student question, "Am I doing this right?" Throughout seatwork Mr. Anderson answered their questions, asked questions, and pointed out their mistakes. Since this was the second day of the quarter, Mr. Anderson's behavior may have been to establish his supportive role, and, as shown by their questions, what students expected of him. Since students could count on Mr. Anderson to tell them whether they were right or wrong they could maintain their absolute ways of knowing.

Mr. Anderson's purpose for seatwork. Mr. Anderson and I discussed his practice of assigning seatwork and walking around to view student work.

> I try to, whenever I have time, to put problems on the board, or assign the problems, for them to work on…. I like it because you can see where each student is at, what they're struggling with, cuz everybody's at a slightly different place, some people need to go back more than this stuff, some people are good with that, you can just go on. (Mr. Anderson Interview, July 5)

His goal during seatwork was to assess students' current understanding so he could adapt his instruction. He added that in the present class he could easily see the work of each student since there were so few (Mr. Anderson Interview, July 5). However, he also valued students' interactions with each other.

> I'll put a problem on the board, or say work on this problem, work together if you want or bounce ideas off each other, or check your answers, convince each other that you're right or wrong, sometimes I use that phrase, but sometimes they still want to get stuck in their little world, they interact with me, as a class, but they don't, you know, they're nervous about sharing with their fellow students… I kind of encourage them to do that because that's something that maybe they can take outside the classroom and, you know, do homework. If they could form a study group it would be beneficial, so that kind of interaction, I try to encourage that in class too, but sometimes I don't get them to. (Mr. Anderson, Interview, July 5).

While he valued students' interaction with each other and believed discussing mathematics and trying to "convince each other" would promote learning and independence, he acknowledged the difficulties in getting students to interact with each other. The difficulties stemmed from students' ways of knowing since absolute knowers believe the teacher has the knowledge and peers can only share what the teacher has provided (Baxter Magolda, 1992). In spite of valuing peer interaction and consistent with his philosophy that he could offer students tools they may not use, he did not require students to work together. In fact, students did not need to work together because Mr. Anderson answered their questions as he walked around. His role as responsive to students' preferences may have allowed students to depend on him rather than build mutual accountability with their peers, and did not contradict their ways of knowing.

Students' preferences for working with peers. Students indicated on their questionnaires their preferences for working with others in the class. In response to the question, *During math class, do you like to work with a partner or in a group? Why or why not?* some students appeared to have interpreted this question as, *If you had a choice between a partner and a group, which would you prefer?* Ten of the thirteen students wrote they liked working with others (See Table 29). Their reasons for wanting to work with others varied: Greg indicated, "easy interactions," so it appeared to be social, while others indicated that students wanted to listen to other students' explanations or ideas. Anthony wrote, "I like to work with a partner in order to bounce ideas off one another; also, there is more of a chance one of the two caught a point in class" (Anthony Questionnaire, June 27). While "bouncing ideas" evidenced Anthony's openness to hearing other students' ideas, the second part of his reply indicated the use of a peer as one who shares the teacher's knowledge, evidence of absolute knowing (Baxter Magolda, 1992).

In contrast, Janet and Thomas indicated that they would benefit if they could explain the math to a partner (Janet Questionnaire, June 27; Thomas Questionnaire, June 27). Thomas wrote, "I like to work by myself because I can understand the material better. Then I like to find someone who is struggling and help them" (Thomas Questionnaire, June 27). While he did not express value for other students' ideas, he realized he may know it better if he can explain it to someone else. He regularly discussed seatwork problems with Kenny, Carol, Sarah, and Kevin.

Three students conveyed a preference for working alone; Kenny, Daniel, and Thomas all wrote that they preferred to work by themselves (Student Questionnaires, June 27), however, during the term all three worked with students nearby. Kenny's location in the classroom made it convenient for Thomas or Janet to ask him questions and it seemed he usually understood what was going on and could explain it (e.g. Fieldnotes, August 3; Fieldnotes, August 16). Daniel regularly asked Carol and Sarah questions or joined in their discussions since they sat close enough to speak without moving their chairs. The day after Mr. Anderson introduced proof by mathematical induction, he assigned a proof as seatwork and Daniel immediately moved his desk to join Carol and Sarah (Fieldnotes, August 16). This move in the last week of the quarter showed he had changed his attitude about working with others over the term. So, providing opportunities for students to work together encouraged students to change their minds about the value of working with peers.

Nature of social norms during seatwork. Most students used seatwork as opportunities to discuss the mathematics with their peers. In particular, Carol, Sarah, Daniel, Thomas, Kevin and Kenny discussed their work often; Anthony and Kathy also worked together whenever they were both present during seatwork. However, the large classroom size allowed some students to sit far from others and therefore these students never interacted with others when I observed (see Figure 3).

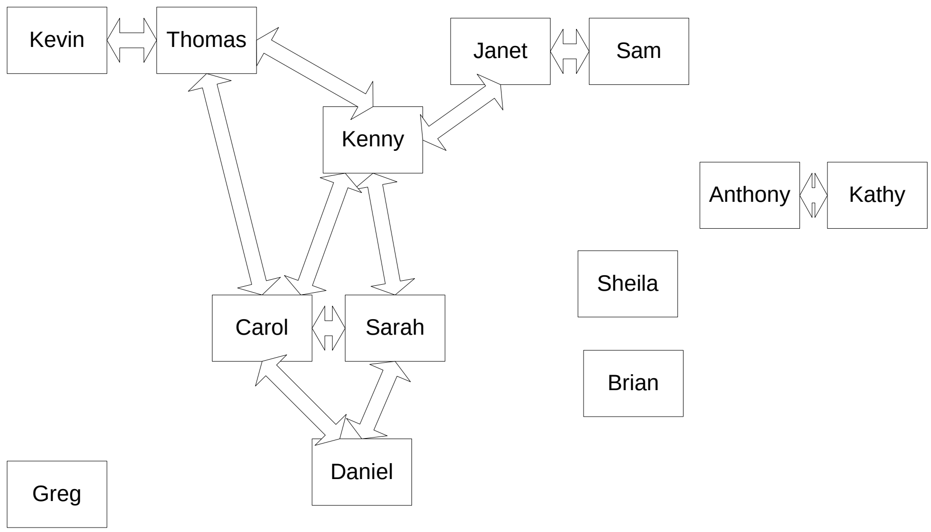

Figure 3. Sociogram of Mr. Anderson's class (not to scale).

Most students worked together and asked each other questions, and some discussed conjectures independent of Mr. Anderson (Fieldnotes, August 9). In this way, Mr. Anderson's class was more similar to the discussion-oriented classes described by Boaler and Greeno (2000) than the didactic classes. Yet, even students who worked together asked Mr. Anderson questions that placed mathematical authority with him (e.g. Fieldnotes, June 28). Since Boaler and Greeno used interviews but not classroom observations, it is not clear that the students who described discussion-based classes in their study did not also rely on their teacher to determine the correctness of their solutions.

As Mr. Anderson walked around during seatwork, his conversations with students resembled communication described by Stigler, Fernandez, and Yoshida (1996) of American elementary classrooms in which the teachers acted more as tutors. The episode given in Appendix L, Lines 29-45, illustrates how Mr. Anderson questioned students, focusing each individual's attention on their assumptions. In contrast, Stigler et al. described teachers in Japanese classrooms using the information they gained from looking at students work during seatwork as opportunities to discuss both right and wrong solution paths during whole-class discussions. Mr. Anderson's responses to students showed he wanted them to make sense of the procedure, but since he focused their attention on their incorrect assumptions, they did not have to use their own sense-making to better understand connections. That is, students depended on Mr. Anderson to determine the correctness of their answers and point out the source of their misunderstandings. In contrast, when he saw their mistakes he could have suggested that they be able to support their answers with another representation such as a table or graph, which may have fostered their ability to determine the correctness of their own work and make sense of the procedures they chose, while shifting intermediate authority away from him.

Developing student-student relationships. In response to: *Do you like to get to know your teacher and/or other students in the class?* most students wrote yes, and those who explained why wrote getting to know others would make them feel comfortable asking questions (see Table 30). However, Kevin and Kathy indicated they would like to know the teacher but not necessarily the other students: "Students – not really. Teacher – of course, he's who I must really learn from" (Kevin Questionnaire, June 27). In spite of his initial response, Kevin regularly spoke to Thomas during seatwork, starting on the second day of the term. During the last two weeks of the quarter, Kevin arrived early to work with Thomas in the hall before class (Fieldnotes, August 9). This presents evidence that Kevin's attitude and disposition towards knowing and learning from his peers changed because of opportunities to work together during class.

Although she indicated on her questionnaire that she liked to work with others (Kathy Questionnaire, June 27), Kathy never spoke to any other student except Anthony while I observed; she knew Anthony before the quarter started since she worked in the daycare his daughter attended (Fieldnotes, July 19). Anthony wrote he wanted to know others but did not want the teacher to force it (Anthony Questionnaire, June 27), indicating that teachers who require students to work with their peers may meet with resistance. Greg was the only student who responded he did not want to know others (Greg Questionnaire, June 27) and he did not talk to anyone except Mr. Anderson when I observed, and only when Mr. Anderson stopped by his desk. Thus, some students may not want a teacher to interfere with the way they intend to participate in the class.

Brian responded, "Yes, it helps me feel comfortable asking questions," to whether he liked to get to know others. However, he rarely asked questions in class other than to request homework problems and he did not get to know anyone in the class, probably because he did not sit near others who wanted to talk and Mr. Anderson answered his questions as he walked around. He also did not show up early for class like several of the others, so he did not participate in conversations between students in the hall. Thus, some students who wanted to know others may not have because it was not expected or convenient.

Sarah's answer indicated she wanted to get to know others (Sarah Questionnaire, June 27) and she moved to sit near Carol on the second day of class (Fieldnotes, June 28). Sarah and Carol worked together each time seatwork was assigned, and Sarah explained how she valued this relationship:

> When you're in a class like this and you don't know anyone and most people who take these level classes, they probably don't know anyone; I think it helps to make a friend. We've exchanged phone numbers and she has a solution manual for the book, so when we were in the last chapter doing the word problems, which I don't like at all, I couldn't figure out one of the problems so I called her up and even if she couldn't understand very well, she explained to me what the solutions manual said. So I just think sometimes two

brains are better than one, and we can work together to try to figure it out. (Sarah Interview, July 13)

Sarah valued being able to call Carol and discuss the problems when she was stuck. However, the nature of these conversations was to convey solutions provided in the solutions manual, rather than discuss their own ideas, instantiating the idea that for absolute knowers, the role of peers was to “share what they have learned from authority figures” (Baxter Magolda, 1992, p. 74). Additionally, Carol regularly asked Sarah questions during class instead of asking Mr. Anderson in front of the whole class, whispering during whole-class discussions or lecture (e.g. Fieldnotes, June 30). This appeared to make Sarah uncomfortable at times, but she still answered Carol’s questions, once by writing and showing it to her (Fieldnotes, July 19). On one occasion when Carol whispered to Sarah, Mr. Anderson paused what he was doing and asked Carol if she had a question. She answered, “Not one that I want to ask” (Fieldnotes, July 7). Mr. Anderson’s question was polite and the pair continued to whisper during discussions throughout the term. I did not see any signs their conversations bothered other students, probably because Mr. Anderson spoke loud enough to be heard over their whispering. Carol’s response to what she does to understand concepts was, “Discuss with others; ask the instructor” (Carol Questionnaire, June 27). Clearly, Carol and Sarah valued discussion with each other as a way to help them learn and used the many opportunities afforded in and out of class.

Sam and Janet sat near each other and both indicated on their questionnaires that they liked working with a partner, and they did occasionally assist each other. However, they did not move their desks together and work in the same way that Carol and Sarah or Anthony and Kathy did. Janet’s reason for working with a partner was that she understood it better once she could explain it to someone else (Janet Questionnaire, June 27). I think the main barrier to their working together was their shyness.

Mr. Anderson offered opportunities for students to develop relationships within the class but did not require students' to get to know others. Students also developed relationships outside of class; Sarah's description of her value of working with Carol was similar to students interviewed by Boaler and Greeno (2000) in discussion-based classes. They described opportunities to discuss the mathematics with their peers, however, they did not say their teachers required them to work in ways they did not want to work. The evidence of this case suggests that while giving students opportunities to work together and get to know each other may change their values of working with peers and are necessary for them to become relational agents, it is not sufficient if their absolute ways of knowing are not contradicted.

Opportunities to know Mr. Anderson. Ten of the thirteen students also indicated they wanted to get to know the instructor (see Table 30). Since Mr. Anderson did not arrive to class early, taught another class immediately after this one, and did not have office hours, opportunities to get to know him outside of class time were limited. However, he provided ample opportunities within class time. Mr. Anderson learned and used students' names early in the term, and walked around to talk to individuals during seatwork. When he walked around on the first day of class, he sat at adjacent desks while talking to some students (Fieldnotes, June 27). There were several times students joked with him during class (e.g. Fieldnotes, June 30; Fieldnotes, July 11; Fieldnotes, July 12), demonstrating that students had become familiar with him. In addition, the slow pace of the class and opportunities to interact throughout each lesson provided opportunities for students to get to know him.

Summary of Research Question One: Development of Roles and Social Norms

Similar to the students in didactic classes interviewed by Boaler and Greeno (2000), students in this class entered with the belief that learning mathematics required memorization, practice, and perseverance. In general, these notions were not challenged as Mr. Anderson provided opportunities for students to participate in ways they were comfortable. However, like the students in the discussion-based classes, they had opportunities to participate in whole-class discussions and group work, and in response some students provided more input than they expected while others interacted with peers more than they intended and in different ways than they expected. However, the opportunities to interact did not necessarily lead students to engage "in the process of validation with the teacher," as suggested by Boaler and Greeno (2000, p. 172) of their discussion-based classes since members of the classroom community expected Mr. Anderson's legitimate role to be mathematical and intermediate authorities of the mathematics.

Roles and social norms also emphasized students' roles as questioners, rather than as providers of ideas, and did not encourage students to listen to each other during whole class discussion. Students' questionnaire responses indicated the roles and social norms of this class aligned with their experiences and preferences of roles in mathematics classrooms. The environment allowed for individual and relational agency if it was consistent with their ways of knowing, since students could persevere and solve problems on the homework, rather than waiting to have solutions presented to them in class and could discuss their ideas with peers during seatwork. However, the roles and social norms assisted students in maintaining their absolute ways of knowing by not challenging their conceptions of authorities as the source of knowledge.

In contrast to recommendations in the *Principles and Standards for School* Mathematics (NCTM, 2000) students were not required to listen to each other and think about the reasoning offered, or attempt to refute or validate mathematical ideas offered by others, behaviors of contextual knowers (Baxter Magolda, 1992). Although Mr. Anderson expressed value for this ideal (Mr. Anderson Interview, July 5), roles and social norms did not foster it.

Research Question Two: Interactions Related to Mathematics

In the following sections I discuss classroom interactions as they related to mathematical activity. Specifically, I examine the nature of communication during whole-class discussion using a framework developed by Brendefur and Frykholm (2000), then consider classroom discourse as it focused on concepts versus procedures, the nature of tasks and their implementation, sociomathematical norms, the role of technology, and how students' contributions influenced the direction of the lessons.

The Nature of Communication

Most of the communication in this class could best be described as uni-directional (Brendefur & Frykholm, 2000). See Table 31 for percent of coding at each level for the first five days of class.

Uni-directional communication. On the first day of class, most of Mr. Anderson's questions required short-answers and the content focused on review and procedural fluency of solving many types of equations covered in intermediate algebra. After discussing the meaning of solving an equation, each example focused on the type of equation and what to do to solve it, such as clearing denominators of rational equations, isolating radicals before raising to powers, and knowing three algebraic ways to solve a quadratic equation.

> Mr. Anderson: Okay, so that's a linear equation. The next type is quadratic [pause]. So a quadratic equation is something that, this linear has power one, quadratic means we're going to have a square running around somewhere. Something like *x*-squared plus nine *x*

plus eight equals zero. [writes $x^2 + 9x + 8 = 0$]. There's a quadratic equation. Now, there are three ways to solve your quadratic equations, anybody remember one of them?

S: Completing the square.

Mr. Anderson: Completing the square, that's one way.

S: Factoring.

Mr. Anderson: Factoring.

S: Zero product property.

Mr. Anderson: Okay, factoring combines to zero product rule. You factor it and then you use - that's excellent terminology - then you use the zero product rule to actually find your answers. So we have completing the square and factoring and using zero product rule, and what else?

Ss: Quadratic formula.

Mr. Anderson: So there's three ways you can do that. Will completing the square and quadratic formula always work? [pause] Factoring only works of course, if it can factor. We'll do one of each type on the board. Anybody knows how this one factors?

S: Eight and one.

Mr. Anderson: Eight and one, yeah; *x* plus eight, *x* plus one. And now we got two things multiplied together gives us zero, we can use the zero factor or product property, whatever they're calling it. So, one of these two things have to be zero, because if two things multiplied together gives you zero only happens if one of them is zero. What if that were ten over here, would that work [indicates replacing 0 with a 10)? [looks around smiling] No, there's no ten-property rule, just a zero-property rule. Okay, so either *x* plus eight is zero or *x* plus one equals zero, so jumping right to the answers [pause], S: Negative eight and negative one.

Mr. Anderson: Negative eight and negative one [circles them after writing them]. We could check them if we want. Let's check negative one, just to try it. Negative one squared, one, nine times negative one… gives us…, okay so it checks out. You can do this with any of these equations, equations you can always check your answer.

(Observation, June 27)

This illustration of uni-directional communication focused on reviewing a procedure for solving quadratic equations by factoring. Closed questions were usually followed by short answers from students, although in one instance above, Mr. Anderson did not wait for an answer. Sometimes, Mr. Anderson interpreted short answers as meaning more than what was said; he repeated and accepted "eight and one" as correct factors of the quadratic expression, but then reworded to give the factors of the expression. While Mr. Anderson discussed a reason for the zero-product property, there was no mathematical reasoning of why there was no "ten-product" property, allowing students to memorize that they must set the product to zero rather than understand why zero but no other number could be used. Thus, this instance of uni-directional communication as it related to mathematics facilitated completion of the procedure and demonstrated that Mr. Anderson would give reasons but did not expect students to give mathematical reasons.

The first day of class was different from any other day I observed in the amount of material covered, the speed, and focus on procedural fluency. However, similar to other observations throughout the term, if the topic was review for students, communication focused on procedures. I discuss more about the focus on concepts versus procedures in a later section.

On some occasions, uni-directional communication became funneling (Wood, 1998); for an example see Appendix J Lines 44-47. This type of questioning did not provide students opportunities to think about what they were doing and why, or to organize their steps since funneling only required them to be able to answer simple questions.

Shifts to contributive communication. Mr. Anderson presented a final example on the first day of class, during which communication shifted to contributive communication:

Mr. Anderson: One more type, absolute value. All right, absolute value of three *x* plus two equals seven [writes $|3x+2|=7$].

S: You're going to get two answers.

Mr. Anderson: Yes, you're going to get two answers.

S: Three *x* plus two is seven and three *x* plus two is negative seven.

Mr. Anderson: [Writes $3x+2=7$ or $3x+2=-7$] All right, now maybe this just comes from a memorized, that is the correct next line, and maybe that just comes from a memorized process. Does anybody want to explain why that line is true? [student has hand up] Yeah?

S(1): Because two numbers make it true.

S(2): Because absolute value is distance from zero.

Mr. Anderson: Okay, because absolute value is distance from zero and, maybe you want to continue off that and say what you were saying about two numbers [points to first student]?

S(1): You're just going to go both ways from zero.

Mr. Anderson: Okay, right, this says I want to be seven away from zero, I could be seven to the right or I could have been to the left. So that's it. Does someone else have something to add to that? [pause] Those are all great. Good stuff. [pause] Maybe one more way for me to think about it is, we're looking for something to take the absolute value of and get seven. Well, what could we do to do that? Well, we could have taken the absolute value of seven or we could have taken the absolute value of negative seven. It's the same thing, just a different way to think about it. (Observation, June 27)

Although this was the first day of class, the social norm that students just speak out before Mr. Anderson asked a question was already established. Students' initial answers were procedural only. However, Mr. Anderson provided students an opportunity to offer explanations; two students answered, and when Mr. Anderson asked the first student to expand, their response was more similar to the second student's answer than their original answer, indicating that they may have listened to the other student's idea. Mr. Anderson expanded on their answers and explained one more way to think about it. Thus, contributive communication focused communication on underlying mathematical concepts and demonstrated that there were multiple ways to think about the concept. In addition, while communication focused on a procedure, the interaction shifted from a focus on instrumental knowledge to relational knowledge. That is, discourse shifted to focus on knowing what to do and why rather than a procedure without reasons (Skemp, 1987). Pesek and Kirshner (2000) showed that students who learned relationally were better able to use the concepts flexibly to solve problems than students who learned instrumentally. The shift also indicated that Mr. Anderson valued understanding why based on concepts.

The following episode shifted from contributive communication back to uni-directional communictions. The class was discussing domains of functions:

Mr. Anderson: Uh, one more type of function you'll see a lot, how about this one [writes $h(x) = \sqrt{3-2x}$]. Root functions, square roots, actually fourth roots, not cube roots or fifth roots, but square roots, fourth roots, sixth roots, you gotta be a little careful because, what?

Thomas: Extraneous.

Mr. Anderson: Yeah, probably, well extraneous would be like the solution that comes about when you're solving an equation.

Sarah: You don't want a negative.

Mr. Anderson: You don't want a negative under the square root. So we don't want *x*-values that make this negative underneath here. So we need to find the *x*-values that keep that positive or perhaps make it zero.

S: Has to be less than two.

Mr. Anderson: Two? Let's see, two. Uh, if I plug in two, well two doesn't even work, because three minus two times two.

Carol: Okay, less than two.

Mr. Anderson: Okay, it definitely has to be less than two, but I don't even think one point nine works. You're thinking about like whole numbers, integers. Yeah, you're right, two doesn't work and bigger numbers than two don't work, just by looking at it, that's how I think you got that, by trial and error, maybe? [Carol nods yes.] We need to find that point though, that actual point, the boundary between numbers that work and don't work, and two's not it.

Sarah: One point five.

Mr. Anderson: One point five is it. How did you get that?

Sarah: Well I just figured that whatever times two has to,

Mr. Anderson: Whatever times two [pause],

Sarah: Yes, has to equal three.

Mr. Anderson: Is three, because if you subtracted it from three then you get, Sarah: Zero.

Mr. Anderson: Zero. So you're like setting this equal to zero?

Sarah: Yes.

Mr. Anderson: And solving? That's what you did, maybe you didn't know that. I want to do a different thing besides setting it equal to zero, though. We really want this stuff in here to be [indicates radicand]? To be able to take the square root, we want it to be either positive or zero, so that three minus two *x* is,

S: Equal to,

Mr. Anderson: Uh, close, do we want it to be greater than or less than?

S: Less than.

Mr. Anderson: I don't think we want it to be less than.

S: Greater than.

Mr. Anderson: We want it to be greater than zero and zero is okay, so greater than or equal to zero. So if you have a square root or fourth root. You guys had the right answer, numbers up to one point five, but a methodology is, you want, take that stuff from underneath, set it greater than or equal to zero; you solve this, one point five or three halves. (Observation, July 7)

Mr. Anderson opened this episode with a closed question and quickly responded to Thomas' answer "extraneous" by explaining how the word is usually used. The communication continued with Mr. Anderson refuting Carol's answer with an example and suggesting what she must have been thinking. Sarah offered the correct answer and Mr. Anderson affirmed it and asked her how she found it. However, while Mr. Anderson perceived he was using a method similar to Sarah's, that a student responded "less than" is evidence they were thinking about the solution set, which was $x \le \frac{3}{2}$, and did not understand what Mr. Anderson was doing with the radicand. This student may still have a process or action view of algebraic expressions (Sfard, 1992) and did not see the radicand as a single object whose value must be non-negative. This exchange demonstrates that during a teacher explanation students may be thinking about their own construction of what is going on rather than following the teacher's way of thinking. While students contributed to the discussion and Mr. Anderson provided time for students to contribute a solution, he maintained his role of intermediate authority by validating and refuting students' suggestions rather than letting other students respond to the ideas. Thus, social norms and roles affected the interaction as it related to mathematics since students could rely on Mr. Anderson to explain clearly and determine the correctness of their answers.

In the following episode, a student's question and Mr. Anderson's response shifted the communication from uni-directional to contributive. The class was discussing a homework problem of finding a polynomial of degree 3 with zeros 1, -2, and 3, and with 3 as a coefficient of the quadratic term (Stewart et al., 2002, p. 278, #49). They began by engaging in uni-directional communication while finding the three factors and multiplying them out to get $x^3 - 2x^2 - 5x + 6$.

> Mr. Anderson: We've actually taken care of this requirement [degree 3, zeros 1, -2, and 3] with the way we set it up. So the only thing left to do is to somehow take care of this requirement [3 as a coefficient of x^2]. We need this coefficient right here to be three. So, let's go back up to this polynomial here, and we kind of talked about this one day where I put other numbers like different exponents and different number multiplication on it and we still had the same zeros. In other words, I can multiply this by seven and I'd still have these two requirements.

Daniel: So, you've got to multiply something through.

Mr. Anderson: Right, you've got to pick your multiplier, so we're not going to pick seven, but we're going to pick our multiplier, whatever it's going to be, we just got to pick it to make this one three. So what are we going to multiply by, negative two by, to get three?

Anthony: Negative three-halves.

Mr. Anderson: Negative three-halves, so there's our multiplier. That's the tricky part on this one, is getting this requirement there. Does that make sense, Thomas?

Thomas: Yeah, it was just that last part.

Mr. Anderson: The last part, I figured as soon as I started going, but oh well, we'll do the whole thing anyways… [Recaps the problem.] Questions on this one?

S: Can you explain how you get the negative three-halves?

Mr. Anderson: How do we get to negative three-halves? Let's see, who said that, Anthony? [Anthony points to the non-participant, who explains what she did.]

Mr. Anderson: Okay, so to cancel out the negative two, that's where this negative two is coming in, and then multiply by three, because that's, yeah, good. Anybody do it a different way? That's great though, that's just seeing it. One way to just kind of see it is, well, I've got to cancel that negative two, and then I gotta multiply by three, making it negative three-halves. Anybody do it a different way?

Anthony: Well, I divided the number we had by the number we wanted.

Mr. Anderson: Okay, so you divided,

Anthony: Negative three.

Mr. Anderson: The number we had by the number we wanted?

Anthony: No.

Mr. Anderson: Other way around?

Anthony: Yeah.

Mr. Anderson: So, the number we wanted by the number we had?

Anthony: Mhm.

Mr. Anderson: Another way, great. You could actually do it equation-wise, if you thought of this as *a*, the multiplier, then our equation would be *a* times negative two equals three. We're going to take the *a*, multiply it by negative two and get three. So, there's three ways, hopefully one of them will help; grab on to one of them. You don't need all three, here. (Observation, July 26)

When a student asked for further explanation, Mr. Anderson provided an opportunity for contributive communication by asking the student who originally gave the answer to explain. This implied he wanted students to listen to ideas from other students, a shift in classroom norms, although he repeated the ideas. Mr. Anderson continued to ask for other explanations then offered a third method, a method within reach of a student who may not "see" what needs to be canceled. He suggested Thomas choose just one method; understanding all three was not necessary. Providing an opportunity for students to share their ways of solving problems suggested there were different solution paths and learners could choose one they understood, which reiterated the idea that there is not always one best way and may encourage the development of students' own voice.

Although Brendefur and Frykholm (2000) described uni-directional communication as either lecturing or closed-questioning, the two formats may allow different opportunities for students. In this class, uni-directional communication was interactive lecture. Since the nature of interactive lecture encouraged some students to speak up with questions and comments, contributive communication was more likely to occur, since students became comfortable enough to share solutions. In a later section we will see that Mr. Anderson used this form of communication to assess students' current understanding and adapt his instruction.

Mr. Anderson also asked open-ended questions that could give rise to contributive communication such as, "ideas?" (e.g. Fieldnotes, July 7; Fieldnotes, August 15). In each of these cases, students needed to organize their thinking rather than find a right answer using cues from the teacher. On some occasions when he asked it this way, he was met with silence, so he reworded his question. For example, after they had discussed exponential functions and the characteristics of their graphs using several examples for the base, he wrote $y = e^x$ and asked, "Can you tell me a little bit about the graph?" He paused, and when no one offered a response, he asked for intercepts, asymptotes, domain, and range (Fieldnotes, August 2). Thus, while Mr. Anderson gave students opportunities to respond to open-ended questions, if students did not provide them, the expectations, roles and social norms of this class provided Mr. Anderson would give a complete answer or ask simpler questions.

Higher forms of communication. As indicated on Table 31, there were no instances of reflective or instructive communication in the first five days of this class, and no instances were observed in later classes. Brendefur and Frykholm (2000) posited that reflective and instructive communication would not occur unless lower forms of communication happened first. However, there is no guarantee that if uni-directional and contributive communication occur, reflective and instructive will eventually follow. In fact, both reflective and instructive communication require students to be willing to share ideas, listen to each other and reflect on each others' ideas, behaviors that indicate contextual knowing. Reflective and instructive communication are unlikely to happen when students do not value each others' ideas and teachers do not suspend their own authority so that students can begin to develop their own voice. In this class, students' expectations of teacher explanations and their absolute ways of knowing, along with Mr. Anderson's acceptance of his roles as intermediate and mathematical authorities precluded opportunities for higher forms of communication.

Focus on Concepts and Procedures

Communication in this classroom could also be distinguished as focusing on concepts or procedures, and discussion of procedures as either relational or instrumental (Skemp, 1987). Procedures included solving equations and inequalities, finding zeros of functions, and using the binomial theorem to expand powers of binomials. Concepts included multiple representations of functions, specifically polynomial, rational, exponential, and logarithmic functions, inverse functions, and composition of functions.

Learning procedures. Communication more often focused on procedures, most of the time with reasons, but sometimes without reasons; procedures were usually connected to concepts when the procedures were new, but when reviewing procedures, communication usually focused on procedural fluency.

Discussion on the first day of class focused on reviewing procedures of solving equations found in elementary and intermediate algebra; communication focused on procedural fluency and recognizing types of equations and how to solve them. For example, to solve the rational equation, $\frac{1}{x-1}-\frac{2}{x}=3$, Mr. Anderson asked how they recognized it as a rational equation:

Mr. Anderson: What do you see in that rational equation that we didn't have in the others?

Kenny: An unknown in the denominator.

Mr. Anderson: Yeah, exactly, the variable in the denominator, the unknown in the denominator, that's the mark of a rational equation. It's not where you want it; you don't want it down below. You need to get it up above. So, how do we get rid of that variable in the denominator?

S: Multiply both sides.

Mr. Anderson: Exactly, multiply both sides by the?

S(1): The *x*.

S(2): The LCD.

Mr. Anderson: Well not just the *x*, we're going to multiply by the *x* minus one as well, or as somebody said it, the LCD. So when I look at it, I find the LCD is *x* times *x* minus one and that's what I multiply by both sides. Multiply both sides by *x, x* minus one. Now when you do that, when you multiply it out, you won't have any variables left in the denominator. (Observation, June 27).

They did not discuss any other method of solving rational equations or even suggest there were other valid methods. This was the only method discussed in the textbook and the reason given in the text for multiplying by the least common denominator was, "to simplify the equation" (Stewart et al., 2002, p. 54). Thus, if students knew other methods for solving rational equations, such as graphing, using tables, multiplying both sides of the equation by each factor in a denominator separately, or cross-multiplying after combining fractions, these methods were not discussed. However, as was his usual practice, Mr. Anderson gave a reason for multiplying by the least common denominator and it was more detailed than the explanation in the text.

Procedures with reasons. Seatwork often provided opportunities to reveal common mistakes in both procedures and concepts. For example, at the beginning of class on the day before the first exam, Mr. Anderson wrote three problems on the whiteboard for students:

1) $\frac{15}{x} - \frac{12}{x-3} + 4 = 0$

2) $a - 2[b - 3(c - x)] = 6$, solve for *x*.

3) $\frac{4x}{x+2} \geq -3x$. (Fieldnotes, July 5)

Students worked at their desks, many together and some alone, for twenty-eight minutes while Mr. Anderson walked around and assisted individuals.

> Mr. Anderson [to Sam]: And then what I do is multiply by the LCD, multiply both sides by the LCD. [Sam responds inaudibly] Yeah, if there's an equal sign, clear fractions. [As he gets to Carol, he looks at her paper] You can't multiply by the,
>
> Carol: I know, because of the inequality, we don't know when it's negative. I wondered when I was doing it. [Mr. Anderson continues walking around addressing this issue]…Okay, you can't lose your denominator.
>
> S: Why?
>
> Mr. Anderson: How did you get rid of it?
>
> S: Multiplied both sides by it.
>
> Mr. Anderson: So what did you do with this? Did you leave it the same or did change it?
>
> S: Left it the same.
>
> Mr. Anderson: How did you know the *x* plus two wasn't negative? (Observation, July 5)

Mr. Anderson's discussion of the first and third problems focused on distinguishing between methods of solving rational inequalities and solving rational equations. While the purpose was to perform these procedures correctly, students were expected to recognize why they could not multiply both sides of the inequality by an expression with an unknown. Of course, there are other procedures to solve rational inequalities that allow multiplication of both sides by an unknown, but the focus in this class was on the standard algorithm discussed in their precalculus text. The textbook presented a list of steps emphasizing isolating zero first, but no reasons were offered (Stewart et al., 2002, p. 81). Thus, the in-class discussion provided more opportunity for relational understanding than the textbook.

Focus on concepts. In some instances, Mr. Anderson focused more on the concepts than the procedures, however, students' responses indicated they were looking for a procedure. For example, when introducing inverse functions, he did not give them a procedure but discussed the concept with a graph and table. The nature of the communication was uni-directional but with a different purpose than the closed questions used when performing procedures. In the following episode he had written $f(x) = x + 4$ and a corresponding table of values.

> Mr. Anderson: Okay, so what we're looking for, ultimate goal of this section, is something called an inverse function. What we think of as a function is, we think of inputting the *x* and getting a *y*. [Pointing to a row in the table] We think of inputting one, our output's five. An inverse function is going to undo whatever we just did. In other words, it's going to go backwards. So we want a function, based on this function here, that'll flip-flop the roles of the *x* and *y*. Our goal is to say, okay, when the input is seven, we're going to get three, when the input is five, we're going to get one. That's what our goal is; our goal is to get this new function that's going to undo whatever we did. So, that's kind of the big picture of what we're looking at here. Let me do one more function…. *g* of *x*, just *x* squared [draws a table]. Now we can do the same thing with

this; we can make a table of values…[He makes a table of values for the function.] Now our big goal here again, is to get these inverse functions, which are supposed to be functions that go the other way. Well, we have a problem with this one if we try to make a function going the other way. Can anybody see the problem with just the values that I've got going on there?

Daniel: You got a negative.

Mr. Anderson: Uh, it's not because it has a negative in it.

Sarah: You don't have negative *y*'s when you square it.

Mr. Anderson: Okay, so that doesn't actually cause a problem if there's not a negative *y* here. I see what you're saying; there's not going to be a negative four on my list, or a negative nine on my list. That just means that negative four and negative nine are not going to be in the domain of the inverse, cuz they aren't going to have that. That's good. But there is a problem, even with these five function values I have on the board. So what we're trying to do,

Carol: Four.

Mr. Anderson [pointing to Carol]: Ah, four is the problem. What happens with four as an input?

Carol: You get two different answers.

Mr. Anderson: You get two different answers. That's not a function. A function is supposed to have, when I give it an input, I get the same output every single time.…[gives two examples from every day life: one a function (buying products in a grocery store), the other not a function (buying cars at a car dealership), then introduces one-to-one functions and horizontal line test, and notation of inverse functions]…and it's supposed to go backwards; whatever our input is here, it's our outputs over there. So, just as some practice points here, here's *f* of *x*, here's a list of these, these are *x*'s, *f* of *x*'s [pointing to the table for $f(x) = x + 4$]. So this also says, for instance, that *f* of one is five, *f* of three is seven [writes $f(1) = 5$, $f(3) = 7$]. That's the same information I have contained on that table... let's try *f*-inverse of seven, just using that function there. Now it's not eleven.

Daniel: It's three.

Mr. Anderson: It's three, yes, what's *f*-inverse of four?

Daniel: Zero.

Mr. Anderson: Zero, what's *f*-inverse of two?

Ss: Negative two.

Mr. Anderson: Where are these coming from? What's happening here?

Daniel: You're solving it backwards.

Mr. Anderson: Yeah, you're just reversing the roles. Over here, where I had the *x*-values and the *y*-values, *x* was our input, *y* was our output. Now we're going the other way around. We're taking what was our *y*-values over here, those are our inputs, those are our *x*-values over here, our *y*-values over here are what used to be the *x*-values. We're interchanging the roles of *x* and *y* when we create this inverse function.

Daniel: So, where does it get hard?

Mr. Anderson: [talks about students' difficulties determining whether a function is one-to-one and proving it algebraically before finding an inverse] All an inverse function does is reverse the roles of, Daniel: You're still solving for x. You still have to, on the seven, you subtract the four over and,

Mr. Anderson: Oh, okay. That's excellent by the way; subtract the four over. Because what I want is, I want the f-inverse of x, I want that as a function rule; f of x is x plus four; I want to know what f inverse of x is, and by the way, you already said it there, you subtract four.

Kevin: So you just gotta switch the sign. If it was negative you'd switch it to a positive?

Mr. Anderson: Well, I wouldn't think of it that way because like what if it was a multiplier, or what if there was more involved than just adding or subtracting? Certainly, if I just have x plus something, the inverse is going to be x minus that same thing. But if I have three times it, it's actually going to be one-third times it.

Daniel: I actually, I just put seven equals x plus four.

Mr. Anderson: Exactly, okay, okay,

Daniel: I didn't do what you just put up there [referring to $f^{-1}(x) = x - 4$].

Mr. Anderson: You didn't do this?

Daniel: No.

Mr. Anderson: Okay, you did, you plugged in the seven where the, [writes $7 = x + 4$] because we had y equals x plus four, plug in the seven for the y?

Daniel: Right, and just solve for x.

Mr. Anderson: Exactly, that's great. But what we want to do now is do that in general. So to do that in general we start with this and let's solve it for x. Because that's what you did, right, you plugged in the seven here?

Daniel: Right.

Mr. Anderson: And then you solved for x, and you got three out of this; that's where this came from. Let's solve this for x in general, which in this case just means subtract four. So y minus four is equal to x [writes $y = x + 4$, then $y - 4 = x$]. This is the inverse function. That's a great way to think about what's going on. (Observation, July 18)

Mr. Anderson emphasized that they needed a function that would reverse the roles, or "undo" what the original function did. As the episode continued, Daniel described a process, "you're solving it backwards," and "You're still solving for x…," while Mr. Anderson's first response focused on the idea of reversing roles. However, Kevin's question indicated he was looking for a rule that would provide a correct answer, "So, you just gotta switch the sign?" Both students appeared to be looking for a procedure rather than an understanding of the concept. Students' goals and ways of knowing during whole class discussion may influence their interpretations of the communication. In contrast, Mr. Anderson emphasized the conceptual ideas of an inverse function rather than provide students a procedure for finding one. Esty (2005) emphasized that a conceptual treatment of inverse functions was necessary for students to be able to use them the way they are used later in mathematics such as to solve equations like $\sin x = c$, and that the procedure of finding an inverse function given in most precalculus textbooks was rarely useful in later mathematics courses. The current precalculus book presented the idea of reversing roles of x and y but followed the presentation with a box containing steps for finding an inverse function (Stewart et al., 2002, p. 233).

Mr. Anderson continued emphasizing the concept of an inverse function the next day when a student asked a homework question: "Use the Property of Inverse Functions to show that f and g are inverses of each other" (Stewart et al., 2002, p. 237, exercises 21-30). Mr. Anderson used $f(x) = 2x$ and $g(x) = \frac{x}{2}$, and composed f and g algebraically, then expanded by discussing what this meant in a way similar to the day before. They started by making a table for each function:

x	$f(x)$
3	6
5	10

x	$g(x)$
6	3
10	5

> Mr. Anderson: We can know this after we know they're inverses even before we look at a function rule. Even if we didn't have this here [indicates rule for *g*, then *f*] or this here, if I tell you *f* and *g* are inverses and you know this point's on *f*, you know this point's on *g*. Does that make sense? Well, we'll see if it does tomorrow; I'll ask that question in words…So if you take something, plug it into *f*, take the answer, plug it into *g*, you're just going to get back to where you started…That's what these two things are saying [he points to the two compositions]. Although this is the technical part, this is how you technically prove that two functions are inverses of each other; it is not what I'm looking for for understanding. However, the understanding of the idea of these pairs reversing, that is what I want you to understand for inverse. And then the extension of that concept: you plug in something into here, you get an answer, you take that answer and you plug it in over here, you're going to get back to what you started with. That's what this is saying. (Observation, July 19)

Rather than providing a procedure to find an inverse function Mr. Anderson emphasized understanding the meaning of an inverse function and focused on why composing a function with its inverse returned the input. The following question was on their exam the next day:

Suppose you know that the point (5, 12) is on the graph of $y = g(x)$. Find a point on each of the following functions (the one that (5, 12) goes to) and give a short explanation of how you found each point; (d) $y = g^{-1}(x)$. (Exam 2, July 20)

As promised, the item required students to explain in words and focused on the meaning of the inverse of a function.

The concept of inverse functions as functions whose *x*-and *y*-values have exchanged roles was also later used to introduce logarithmic functions as inverse functions of exponential functions using multiple representations (Observation, August 2). The connection was maintained as they discussed the characteristics of the graphs of logarithmic functions, and found function values and properties such as domain, range, asymptotes, and intercepts.

While many of the procedures were presented as if there was only one correct way to work them, Mr. Anderson explained reasons for the procedures at least as deeply as the textbook, and sometimes expected students to use the reasons when they worked exercises (e.g. Observation, July 5). Lobato et al. (2005) described teacher explanations with the intention of prompting students to make sense as initiating, and reported that initiating combined with eliciting to see how students interpreted the information could aid in conceptual development. See Appendix L for an example of an initiation followed by an elicitation; Mr. Anderson presented a problem which Kenny quickly answered. Mr. Anderson elicited by giving students a similar problem to see if they had made sense of the explanation. Lobato et al. pointed out that if the social norms of the class allow students to reproduce the teacher's idea rather than respond with their own, the elicitation would not work to aid conceptual development. In this episode, students did not appear to understand that Kenny used the idea that if the point is on the graph of the function, he can replace *x* and *y* in the function with the values of *x* and *y* at the point to find *a*. Several student instead replaced *x* with 2 in the second problem because *x* was replaced by 2 in the first problem, rather than use the *x*-value that corresponded to the *y*-value they were given. These students tried to reproduce the procedure without making sense.

The Implementation of Tasks

Mr. Anderson provided opportunities for students to work on tasks by assigning seatwork, often from textbook problems (see Appendix M for text problems assigned as seatwork). I also discuss homework problems as tasks since they were assigned for students to work and could be categorized by features such as multiple representations, multiple solution paths, communicating, and reasoning (Henningsen & Stein, 1997) and categorized by cognitive demand. The cognitive demand of task features varied from memorization to “doing mathematics” (Stein, Grover, & Henningsen, 1996).

Declining cognitive demand. Tasks introduced as seatwork included problem solving, and applications stressed interpretation. For example, Carol and Sarah worked together to find a linear model for the cost to build chairs, while Daniel worked alone but checked his answer with them (see Appendix M, section 1.10 #70 for a statement of the problem). They had found an equation to model the cost to manufacture chairs: $y = 13x + 900$ when Mr. Anderson approached:

Sarah: The thirteen is what it costs to make one chair?

Mr. Anderson: Exactly, that's what it costs to manufacture or make one chair. So what does the nine hundred mean?

Carol: I don’t know.

Daniel: Where you break even?

Mr. Anderson: No, let's think, if we make zero chairs, we'd still have to pay nine hundred dollars. What would you have to pay for?

Carol: Your store, it's your plant.

Mr. Anderson: Yeah, maybe rent, maybe there's some employee you can't lay off, you know, there's some manager who always gets his salary, some electricity you pay for whether you make a chair or not.

Sarah: It's the same, [inaudible] number of things.

Carol: Cost not directly related to producing chairs. (Observation, July 5)

Consistent with the roles and social norms of this class, Sarah's original question asked Mr. Anderson to validate her answer, which he did, but also followed with a question to interpret the *y*-intercept, increasing the cognitive demand. Carol did not appear to have a way to think about what the nine hundred meant, while Daniel may have had cost functions associated with a break-even point, but was not thinking about the meaning of the function they had derived. Mr. Anderson's next question used the meaning of the function, "if we make zero chairs, we'd still have to pay nine hundred dollars, what would you have to pay for?" decreasing the cognitive demand of the task. Students did not have to understand how their function modeled the situation to answer Mr. Anderson's question. In general, while Mr. Anderson asked questions to help students continue solving the tasks, the nature of roles and social norms during seatwork ensured students would have a correct answer after Mr. Anderson stopped by their desks.

The task in the following episode began as seatwork and can best be described as "doing mathematics" (Henningsen & Stein, 1997). After Mr. Anderson introduced sequences and partial sums; he asked students to find the third, seventh, and hundredth partial sums of the sequence 1, 4, 7, 10, 13, 16, 19....:

Mr. Anderson: Okay, let's go back to one of our earlier sequences. Uh, *a* sub *n* is going to be three *n* minus two. Find *s* sub three, *s* sub seven, and *s* sub one hundred. …. This one's a challenge [indicating hundredth partial sum]. You should be able to get these two. [Mr. Anderson walks around while they work; he also writes the first eight terms on the board.]

Daniel: You have to have a formula to find one hundred.

Mr. Anderson: Yes, you're going to try to get a formula. Now, if you got *s* sub three and *s* sub seven and you got those pretty easy but you're having trouble with the formula, go back and do a couple more, like *s* sub four and *s* sub five, and try your formula out on those ones. Okay, so I'm giving you an idea on how to do the hundredth one. [It is quiet for 18 seconds.]

Kenny [to himself but out loud]: Huh, I'm getting lost. [The class is quiet as Mr. Anderson walks around for twenty more seconds looking at each paper.]

Mr. Anderson: Okay, this one was a challenge, if you don't get the one hundred, that's okay. I'll come up with something on the board. You should at least get *s* sub three and *s* sub seven [quiet for 27 seconds]. Okay, let's go with *s* sub three?

S: Twelve.

Mr. Anderson: *s* sub seven?

S: Seventy.

Mr. Anderson: Questions on either of those? So we got those okay? All right, maybe before we do *s* sub one-hundred, let's do an easier one that's kind of similar to this one, and then we'll come back and do *s* sub one-hundred. So, here is a similar but easier problem; I want the sum of all the numbers from one to one hundred. You guys go, 'wait a second, that doesn't seem easier.' Well, it's easier to see what we're going to do. So, what if I paired numbers starting from the outside? So I'm going to pair one and one hundred, and add them up, what do I get?…[they find the sums of several pairs are each 101]…Is there a pattern going up here? When I'm pairing from the outside in, I'm getting the same sum every time, of one hundred and one. So how many times am I going to get one hundred and one?

Anthony: Fifty.

Mr. Anderson: Fifty times, I stopped here at a hundred; I didn't keep going [responding to a student who indicated infinite]. See if I kept going, yeah. Fifty, there's fifty pairs I'm going to get, that are all a hundred and ones. So this sum, if you add it all up is, fifty times this, is five thousand fifty…Back here, did we have the one-hundredth term written down somewhere?

S: Two ninety-eight.

Mr. Anderson: What was it? Two ninety-eight? So if you use that same idea over there, the pairs are going to be one and two ninety eight, which is going to be two ninety-nine. How many pairs are we going to have?

S: Fifty.

Mr. Anderson: Fifty, yeah…So let's take two ninety-nine times fifty is… [Finishes writing answer and tells a story about Gauss.]

Daniel: To get two-ninety eight, you took a hundred times three and subtracted two?

Mr. Anderson: Right.

Daniel: So then why, why did you then go to two ninety-nine?

Mr. Anderson [to the class]: Where did the two ninety-nine come from?

Anthony: The first term.

Carol: Yeah.

Mr. Anderson: Yeah, now I'm pairing the first,

Daniel: Oh, okay.

Mr. Anderson: And the hundredth, so one plus two ninety-eight is two ninety-nine, four plus two ninety-five is two ninety-nine, seven plus two ninety-two is two ninety-nine. Does that make sense? Good question.

Kevin: So we took fifty times two ninety-nine?

Mr. Anderson: Yeah, so we took fifty times two ninety-nine to give us fourteen thousand nine hundred and fifty. [Next they used the same method to make sure it also worked for the seventh partial sum.] (Observation, August 15)

The task as written and set-up was "doing mathematics" because students were not given a method but expected to systematically explore and solve the problem. In fact, in response to Daniel's assertion that they needed a formula, Mr. Anderson gave a suggestion on how to continue looking for a pattern to find a formula. But after students showed frustration, he said he would "do something on the board," excusing them from trying any further. Students appeared to have no experience implementing tasks at this level and Mr. Anderson initially hinted it would be difficult, similar to the low-press teachers studied by Kazemi and Stipek (2001). Mr. Anderson then scaffolded by doing a similar problem, making connections which most students appeared to follow. At this point he could have asked them all to work the original problem in their groups or individually, and to be able to explain the mathematical reasoning behind their work, and if students understood the simpler problem, they could have completed the task as "procedures with connections." However, Mr. Anderson did the problem by asking closed questions, allowing the task to decline completely as he did the problem.

This was the only class day reserved for discussing sequences, partial sums, and introducing proof by induction, so lack of time and familiarity with the new concept of sequences were factors contributing to the decline in cognitive demand. Others factors included lack of expectation by Mr. Anderson that students could do the task, and students' lack of perseverance, similar to middle school classrooms studied by Stein et al. (1996). In addition, rather than ask students to contribute what they did and what they learned from it, then use their ideas to finish solving the problem, Mr. Anderson used his ideas. Using students' ideas while silencing their own is one way teachers can foster students' development of more complex ways of knowing (Baxter Magolda, 1992).

Students' expectations of tasks. Students' questionnaire responses did not indicate they had experience working on tasks that could be described as "doing mathematics," although some students expressed a desire to work problems in class. For example, Kenny's goal was to "attempt to work problem before instructor does" (Kenny Questionnaire, June 27) while Sarah and Carol wanted to be able to practice during class (Carol Questionnaire, June 27; Sarah Questionnaire, June 27). These answers indicate students wanted to practice doing procedures they had been shown how to do so they could either check their answers or get feedback from the teacher.

Some of the tasks originated in students' homework and were brought into class discussion by student request. The nature of the homework tasks was typical of college precalculus textbooks; there were examples in the sections for the easier exercises, but later problems required students to make connections and extend the ideas. Daniel commented on this to Mr. Anderson, saying that the problems were tricky and there were no similar examples in the text (Fieldnotes, July 5). However, Mr. Anderson responded:

> No it doesn't try to trick you. See precalc is starting to, instead of showing you everything, it’s starting to make you think, so, the further you go in math the more it happens where they show you something and then in the homework you have to extend those. (Observation, July 5)

While Daniel’s comment indicated his belief that mathematics consisted of procedures and students must be shown how to do each type, Mr. Anderson suggested students must make connections and extend what they have learned. Mr. Anderson intended for students to struggle with problems they had not been shown how to do; he reported it was a teaching strategy of his to introduce new material with examples that were fairly easy, then let students work on the homework which would contain problems harder than the ones worked in class, followed by discussion of them the next day as students requested (Mr. Anderson Interview, August 9). Thus, he did not try to give students examples to follow for each type of problem so they could reproduce procedures.

In one notable episode (see Appendix J), the textbook problem required students to use an idea that had been focused on repeatedly in this class, that of finding a function value from a graph, and use it to find values for compositions of functions. When he asked Mr. Anderson the homework question, Daniel admitted he did not have any idea what the question meant. Mr. Anderson invited the whole class to join them in working this problem and began by focusing their attention on what information they could find from the graph, after which Daniel indicated he knew how to do it. However, Mr. Anderson offered to do the first problem and, as students asked for more, finally did them all. This classroom dynamic was similar to that described by Stein et al. (1996), in which students pressured teachers to reduce task complexity, and similarly reduced the cognitive demand. Mr. Anderson focused on the meanings of the objects, however, his intention to be supportive reinforced students' ideas about their roles and the nature of learning mathematics as reproducing procedures first modeled by the teacher, ideas characteristic of absolute knowers.

Because of the roles in this class, Mr. Anderson solved many tasks during whole class discussion that had been assigned as homework problems. Students had the opportunity to work the problems on their own first, so some students may have spent time trying to figure out a problem, or may have decided quickly that they did not know how to do it and asked in class the next day since Mr. Anderson answered all homework questions completely. Thus, the value of homework problems as tasks was limited by the social norms and roles of this class and students' ways of knowing.

The Influence of Student Contributions

Students used the many opportunities afforded them by Mr. Anderson's wait-time and encouragement to ask questions and make comments during whole-class discussions and seatwork. This section examines the nature of students' input and Mr. Anderson's responses and the effects of students' questions and comments on the lessons.

Nature of students' questions. While not frequent, some student questions asked about expectations, "will there be one like that on the exam?" (Fieldnotes, June 28). Other examples include, "On the test do you want us to work it out or can we just leave it like that?" (Fieldnotes, July 19) and, "What happens on a test if that's what I wrote down?" (Fieldnotes, June 28). Mr. Anderson did not respond directly to questions about what would be on exams, but worked the problems in whole-class discussion (Fieldnotes, July 19).

Other questions from students could be coded in one or more of the categories: seeking a rule, seeking an explanation, checking their thinking, or expanding the discussion. The first three categories describe students' purposes for asking the questions, while the fourth category describes the nature of Mr. Anderson's response, so some questions could be coded in more than one category. For example, if by checking their thinking, discussion was also expanded, it was coded in both categories.

Checking their thinking. Student questions coded at *checking their thinking* were often requests for validation as Mr. Anderson walked around during seatwork. However, some questions of this type could also be considered conjectures. After Mr. Anderson presented the class with a fifth-degree polynomial and asked them to find the zeros, Daniel interjected, "Does the *x* to the fifth tell you there's going to be five?" (Observation, July 25). This idea had not yet been introduced, but Mr. Anderson discussed it in response to Daniel's question as they found the zeros. Similarly, in the following episode Sarah's question appears to be a conjecture; however Mr. Anderson's response focused on the meaning of "even" instead of the conjecture. Students in the class had just used their graphing calculators to find the extreme values of $h(x) = x^4 - 5x^3 + x$. Mr. Anderson had introduced even functions earlier in the lesson, relating the concept to symmetry, which they had discussed previously in the quarter.

Mr. Anderson: Questions on that?

Sarah: Will you only have an absolute minimum if it's an even function?

Mr. Anderson: Yes.

Sarah: Because I was thinking that it's kind of like that [Sarah continued talking but it was inaudible because Mr. Anderson talked over her to correct himself].

Mr. Anderson: Well not an even function, because this is not an even function, even highest power [he continued with an explanation of the difference between an even function and a polynomial with even highest power]. Other questions? (Observation, July 13)

Sarah's question indicated she expected Mr. Anderson to validate her thinking, but because of her misuse of the word "even" the focus became the distinction between even functions and functions of even degree, rather than the intent of her question. The class could have investigated Sarah's question as a conjecture, and the distinction between even functions and even highest power most likely would have emerged during such a discussion, but it was not the social norms of this class. Rather, Mr. Anderson's role was to clearly explain any misconceptions he perceived, acting to "[try] to clear up confusions as quickly as possible" (Chazan, 2000, p. 117).

Other instances of *checking their thinking* involved students making connections. For example, while finding zeros of polynomials with real coefficients, Mr. Anderson mentioned that non-real complex zeros always come in pairs. Daniel asked "Is that because of the quadratic formula?" (Fieldnotes, July 27). Daniel realized that the quadratic formula resulted in two solutions when the discriminant was negative and connected the result of the process with Mr. Anderson's statement. Similarly, after the class found complex zeros of a polynomial function, Carol asked, "How come that doesn't show up on the graph? Because of the *i?*" (Observation, July 26). Both students appeared to be striving to make connections and wanted to check their thinking with Mr. Anderson. The social norms that students should ask a question or make a comment whenever they wanted to provided opportunities for more connections to be discussed.

Seeking an explanation. Student questions coded *seeking an explanation* were often requests for the teacher to re-explain, indicating the student was still trying to understand the idea, but other times they were requests for something new. For example, at the end of a class period, Mr. Anderson told them there were several parent functions they should be familiar with and directed them to look in their textbooks:

> Carol: What's greatest integer?
>
> Mr. Anderson: Oh, okay, let's look down there at that last one. That probably is a new one. All the others you can get through point plotting if nothing else. So, let's look at *f* of *x* is the greatest integer function. This means find the greatest integer less than or equal to *x*. So, the words here are find the greatest integer less than or equal to *x*. Let's try some, like what's *f* of three point seven one? So, we need the greatest integer that's smaller than or equal to three point seven one. So, think like whole number.
>
> S: Four.
>
> Mr. Anderson: So, four, four is the closest integer, but four is not less than three point seven one. It's not four [pause]. So, I need integers less than three point seven one. Just tell me some integers that are less than three point seven one.
>
> S: Three [several students said it right away].
>
> Mr. Anderson: Three, two, one, negative seventeen, negative five hundred. What's the largest out of all of those?...[continues to give more numbers to evaluate the function including irrational and negative numbers; uses a number line.]…Now, if you were to graph this function, you get that picture on the bottom right on page one seventy four, you get these flat lines, and then you jump up a level, then go straight for a while, then you jump up to the next integer. So it's also called, sometimes it's called the step function.
>
> Carol: Is it always a closed circle at the left hand side?

Mr. Anderson: It's always a closed circle at the left hand side and an open circle at the right hand side. And that has to do with, at five you're going to get five, at five point one two you're going to get five, at five point nine, nine, you're still going to get five, but when we get to six, it's not at the same level anymore, at six you jump up to the next level.

Carol: I get it, okay.

Mr. Anderson: Actually, step functions are very useful for modeling,

Carol: Where?

Mr. Anderson: For instance, phone airtime, is a step function. They take the nearest minute, but they don't round down.

Carol: They go up.

Mr. Anderson: They go up, so they may be the least integer greater than. It's a very similar thing, so a lot of things are modeled by step functions. (Observation, July 7)

When asked about the greatest integer, a question clearly aimed at eliciting an explanation from the teacher, Mr. Anderson provided a definition and then asked questions to give students an opportunity to use the definition. He refuted a wrong answer with the reason and continued supporting understanding of the idea by having students evaluate the function at several different values and drawing a number line to help them think about it. Carol asked about the endpoints of the intervals, apparently seeking a rule, but Mr. Anderson's response included the reason. There was no indication Carol tried to answer her own question before she asked it, although from observing her I believe she was capable of answering it. The nature of this question and Mr. Anderson's willingness to respond maintained his intermediate authority as the expert who was able to understand and explain, so did not challenge students' absolute ways of knowing.

However, he also used this opportunity to connect students to an unfamiliar function by describing a common use for it, a strategy that advances students' ways of knowing (Baxter Magolda, 1992).

Seeking a rule. Students commonly asked questions indicating they were seeking a rule. In the following episode, several students asked questions indicating they wanted a rule. Mr. Anderson introduced piecewise functions and the example, $f(x)=\begin{cases}2x-3, & \text{if } x\leq 2\\ -3x, & \text{if } x>2\end{cases}$.

> Mr. Anderson: What happens in-between them, do they connect? Should we connect them? [Sarah shakes her head no] No, we don't actually connect these lines.
>
> Sarah: One of the lines should have like an open?
>
> Mr. Anderson: Yeah, there should be open circles somewhere. Well, okay, let's look, which one is going to get an open circle and which one, this one, it includes that endpoint, two, one. So that looks good, which means there probably should be an open circle down here, somewhere. Now the question is where should that open circle be? Well, here's a little trick you can sometimes use: what if we did plug two in here? I know that for the *f* function, we're supposed to put two up in there. But, what if we did plug two in here? Then we'd get negative six, so at two, negative six, that point though is not really there, though, so that's why you're going to get an open circle. There, there's our graph. But if you plug in two point one for instance, we're supposed to get negative six point three, or even if you plugged in two point zero, zero, zero one, you'll get negative six point zero, zero, well, however many zeros, three, but you're never going to get that negative six, though. Questions on that?
>
> Sarah: So if you had three piecewise, would you have three separate lines?

Mr. Anderson: Yeah, potentially, well, they don't have to be lines, too, like we had that first one that we did, *h*, yeah; it could have had a squared part to it. But they could be three disjoint things. Now, sometimes they connect at that place so it doesn't even look like there's an empty circle there…[gives an example]. But yeah, you could have three, four, five pieces, whatever.

Thomas: Will that trick for the open circle work for all of them or just this one?

Mr. Anderson: I don't know if it will work for all of them because, um, like some could be undefined at that place, like if we have, I don't know that it works for all of them. You could always try and see. So what we're talking about here is, two doesn't really work in this one, but what if we plug two in? Does that work all the time? And, I don't think it works all the time, but you could always try it, you know, plug that value in, if you get a value, most likely that's the continuation of it. Other questions on this?

S: So, when we're doing our tables we could just do the five points close to negative one?

Mr. Anderson: Well on the table, for this one, what would be really important, well looking right here, it's really important, close to two, to try to figure out what happens. So, it looks like I ought to do two point one and I ought to do three. I get a lot of values close to the point where it changes for me, personally, to kind of get an idea…. If I were doing this calculator-wise [explains how to use graphing calculator to graph the function] … So you can use your calculator here to help you out.

Kevin: So we can just graph the line and draw it for those conditions?

Mr. Anderson: Yeah, exactly, you can just graph this line, and say okay, now let's just pick the appropriate *x* value points on it. (Observation, July 7)

Several students, including Sarah, Thomas, Kevin, and another student all interjected questions meant to generalize procedures, and appeared to be seeking rules. Sarah's question focused on what they should expect the graph of a piecewise function to look like, while Thomas wanted to know if the "trick" Mr. Anderson suggested worked in all cases, and Kevin wanted to know if they could always use their calculator to graph piecewise functions. The other student was trying to find a rule on which values to input before attempting to graph. This provides evidence that students were concerned with finding methods to ensure correct answers. In each case except Kevin's, Mr. Anderson responded by explaining more analysis needed to occur, he could not say whether or not it would always work. However, he focused more on the topics addressed by the questions in later episodes, helping to clarify the distinctions.

Expanding the discussion. Mr. Anderson addressed Sarah's issue a little later on the same day by assigning seatwork to graph $f(x)=\begin{cases}|x|, & \text{if } -3\le x\le 3\\ -2x+9, & \text{if } x>3\end{cases}$. As he walked around and engaged in individual conversations, Mr. Anderson's suggested which function values to find and focused students' attention on reasons for closed and open circles (Observation, July 7). This seatwork related to Sarah's question about having as many disjoint graphs as "pieces" of a piecewise-function. Mr. Anderson focused their attention on the fact the pieces connected and challenged generalizations students made about whether there was an open or closed circle on the graph. In general, students' questions provided Mr. Anderson with information on their current understanding, and he used this information to introduce new problems and examples targeting the ideas in the questions, a component of instructive communication. Striving to understand and work from learners' previous constructions and using examples to introduce perturbations are valued by teachers subscribing to constructivism (Ernest, 1996). While Mr.

Anderson did not claim to subscribe to constructivism, he did want to know what students were thinking so he could adapt instruction.

There is evidence Carol understood the "trick" of finding where to place an open circle and was able to use it later in the quarter when they graphed a piecewise function:

$$f(x)=\begin{cases}1-2x, & x\le 0\\ 2x-1, & x>0\end{cases}.$$

> Mr. Anderson: Now what happens, though, as we get closer to the *x*-values, um, closer to zero? [He's indicating closer to zero from the right.]
>
> Carol: It's an open circle at zero, negative one.
>
> Mr. Anderson: Okay, it's an open circle, where at?
>
> Carol: Zero, negative one.
>
> Mr. Anderson: Yes, how did you figure out exactly where that was at?
>
> Carol: I know what I do, but I don't know if it's the right way.
>
> Mr. Anderson: That's okay, just tell me.
>
> Carol: I go ahead and plug zero in even though it's not equal to.
>
> Mr. Anderson: Yeah, even though zero doesn't really work, go ahead and plug it in.
>
> Carol: All the pieces between zero and one do fit, so you gotta go all the way to the line.
>
> (Observation, July 19)

Carol used the "trick" and understood why it worked and what it meant as evidenced by "all the pieces between zero and one do fit…," although she did not think it was the "right" way to find the endpoint. Carol still believed there were certain right and wrong ways to do problems and did not trust her own sense-making as a valid way to justify mathematics. Although there was other evidence Carol did not think she was being mathematical unless she used a procedure, the terminology Mr. Anderson used, "trick," may have added to that belief. Instead, he could have indicated that sometimes there is no procedure, but that each situation can be analyzed, and presented the trick as a tool for analyzing function behavior near an x-value.

At other times Mr. Anderson used opportunities presented by students' questions to clarify ideas. When Daniel checked his thinking by asking about his interpretation of his answer in a reduced matrix, Mr. Anderson used it as an opportunity to ask about two other cases:

Daniel: On thirty-nine, on my end matrix, the third row, I've got zero, zero, zero, zero, zero. So is that just zero equals zero?

Mr. Anderson: And, does that tell you any new information?

Daniel: I wasn't sure.

Mr. Anderson: Okay, so that last row, you've got. So my question is, [writes a row of zeros] that last row, does that tell us any new information?

S: No.

Mr. Anderson: No, we already knew zero equals zero, irregardless [sic] of that.

Daniel: Okay.

Mr. Anderson: So, you get to ignore the last row. On that one, on thirty-nine, since we have four variables, that means you're going to get two of them with parameters since the last row is zero [pause 10 seconds]. What if it said this though? [changes last row to $\lfloor 0 \quad 0 \quad 0 \quad 0 \quad 1 \rfloor$.]

Ss: No solution.

Mr. Anderson: No solution, now it says zero equals one. What if it said that? [writes $\lfloor 0 \quad 0 \quad 0 \quad 1 \quad 0 \rfloor$].

S: Then z would equal zero.

Mr. Anderson: Yeah, *z* would equal zero, whatever variable this was would equal zero. Good questions. Other questions? (Observation, August 15)

Daniel wanted reassurance he was interpreting the result correctly and Mr. Anderson provided a reason for the interpretation. Then Mr. Anderson took the opportunity to ask two more questions, focusing on the meaning of the rows. The episode consisted of uni-directional communication, however communication focused on students' current thoughts. Daniel's need for reassurance demonstrates while he may have had an idea about the answer to his question, his evaluation of whether he was right or not rested on Mr. Anderson's agreement.

As previously discussed when explicating the social norms of this class, rather than ask a question about a specific concept or procedure, students sometimes requested examples of a specific type. In some cases, Mr. Anderson used the request as an opportunity to choose an exercise that was problematic. When Daniel asked for a transformation problem that included absolute value, Mr. Anderson chose $y = 2 - |x|$ (Stewart et al., 2002, p. 195 #40). The first student answer to the effect of the two was "stretch." The day before, Mr. Anderson had emphasized the idea of a *factor* stretching the graph vertically, anticipating students' greater attention to the placement of numbers than to the operations (Fieldnotes, July 12). If Daniel had not asked for an absolute value example, this issue would not have arisen. However, Mr. Anderson capitalized on the opportunity to present a problematic task.

Mr. Anderson listened to students and produced examples based on what their comments or questions indicated about their thinking:

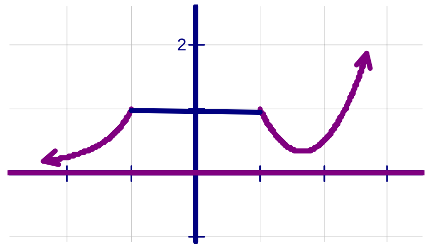

Figure 4. Domain of a function given in graphical form.

Mr. Anderson: What's the domain and range of this guy [see Figure 4]?

Daniel: Negative infinity to,

Ss: All real numbers.

Mr. Anderson: All real numbers for the domain.

Daniel: Same with the range.

Kenny: Zero.

Mr. Anderson: Okay, now let's look at this arrow right here.

Daniel: It's going to keep going.

Mr. Anderson: Yeah, it's going to keep going, and it's going to keep getting lower there, but it's a different arrow than like this arrow and this arrow [points to both arrows on the graph]. See, this arrow is indicating we're going down forever; this arrow looks like we're going up here.

Kenny: That arrow's [getting closer to] the *x*-axis.

Mr. Anderson: Right, but this arrow [points to the left] the way I've kind of drawn it there shows that we're never crossing the *x*-axis. But we're continuing to get closer and closer to it. So, actually for our range here, zero to infinity, and we don't actually ever get to zero.

S: We're staying above it... [discussion continued].

Mr. Anderson: Let's do that for our last graph; *f* of *x* equals one over *x* squared.

[Mr. Anderson demonstrated how a function may continue to shrink but never reach zero by substituting values of x into the function.]

Mr. Anderson: When x is a million, one over x-squared is a tiny number but it will still be positive. It will keep getting smaller and smaller and smaller. (Observation, July 11)

In response to Daniel's misinterpretation of the meaning of the arrow, Mr. Anderson presented an example whose graph behaved similar to the previous example but students could see from the symbolic form how large values of x resulted in function values that continued to decrease but never reached zero. This decision was consistent with Mr. Anderson's perception of his own role of wanting to know what students' confusions were so he could help them understand (Mr. Anderson Interview, July 5). That is, assessment played an important role in guiding Mr. Anderson's actions. "The *assessment-centered lens* encourages the need to provide frequent opportunities to make students' thinking and learning visible as a guide for both the teacher and the student in learning and instruction" (Donovan & Bransford, 2005, p. 13).

In summary, the social norms and roles of this class afforded students many opportunities to ask questions and make comments, which provided opportunities for Mr. Anderson to assess their understanding and adjust his instruction. But the nature of student input reflected their positions as absolute knowers, and Mr. Anderson's responses as intermediate authority may have helped support their positions. Social norms and roles constrained communication to lower levels of uni-directional and contributive. Mathematics educators suggest much more engaged and diverse roles for students in discourse, including trying to convince themselves and others of the validity of their mathematical ideas (NCTM, 1991).

The Nature of Sociomathematical Norms

Sociomathematical norms affected by social norms. In Mr. Anderson's class the social norms allowed ample opportunity for students to participate by asking questions or making comments. However, it was not a social norm that students explain their solutions or thinking. A sociomathematical norm related to whole-class discussion is that students' explanations of procedures should be based on concepts (Yackel & Cobb, 1996). As discussed in earlier sections, students and Mr. Anderson agreed it was Mr. Anderson's role to explain and their roles to ask questions and those roles limited possibilities for this sociomathematical norm of students' explanations to develop. While Mr. Anderson asked students how they found their answers (e.g. Fieldnotes, July 7; Fieldnotes, August 3), students often responded by stating procedures too short or too vague to be useful for other students' understanding. As discussed in earlier sections, Mr. Anderson acknowledged students' answers by repeating them and explained for the rest of the class.

The social norm described above supported a sociomathematical norm that mathematical ideas were validated when Mr. Anderson affirmed them. In the following episode from the fifth day of class, the purpose of Mr. Anderson's questioning appeared to be to find out what students already knew about functions and to focus their attention on the idea that if $f(a) = b$, then (*a, b*) is an ordered pair on the graph of $y = f(x)$, an idea he focused on several times throughout the quarter. After introducing functions, he wrote: $f(x) = x^2 + \frac{1}{x}$.

> Mr. Anderson: We can do things like evaluate it and find *f* of one. How do we find *f* of one?
>
> Kenny: [He raised his hand before saying it] Put one in for all *x*-values.

Mr. Anderson: Right. Put one in for all the *x*-values. So it'll be one squared plus one over one, it'll be one plus one, which is,

S: Two.

Mr. Anderson: Two. What kind of information is contained in this? [points to $f(1) = 2$]. There's a lot of information in there. What kind of information do we get out of this?

Kenny: An *x*-value and a *y*-value.

Mr. Anderson: Okay, which one's the *x*-value?

S: One.

Mr. Anderson: Which one's the *y*-value?

S: Two.

Mr. Anderson: Okay, so we have an *x*-value and a *y*-value. So, what we're really talking about is an ordered pair one, two. Now, what does that one, two, as far as,

S: It's a point on the graph.

Mr. Anderson: Okay, yeah, it's a point on the graph of this function. That's exactly what it is. (Observation, July 7)

They continued with the lesson. Then much later in the lesson Mr. Anderson drew a graph on the board:

Mr. Anderson: All right now, from this graph, let's see if we can find some things: like what is *f* of negative three? I don't have a function rule to plug it into, I just have a graph.

Kenny: Three.

Mr. Anderson: It's three! Excellent. How'd you get that Kenny?

Kenny: *f* of negative three is three.

Mr. Anderson: That's true, *f* of negative three is three; how'd you get that, go ahead.

Kenny: Because, you said before that that's your *y*-value [OC: Kenny seems to be uncomfortable with the pressing].

Mr. Anderson: Yes, right this is the *x*-value, this is the *y*-value, so how'd you get that from this graph?

Kenny: It's a point.

Mr. Anderson: Okay, yeah, there's this point over here, negative three, three. That point's on the graph; that means that has to be the function value. (Observation, July 7)

Mr. Anderson did not expect students to easily find function values from a graph. He pressed Kenny to make the connection explicit for the rest of the class, but Kenny's responses indicated that while he was comfortable in the role of giving a right answer, he was not comfortable in the role of expanding on his answer. Kenny also said the point gave *x*-and-*y*-values because Mr. Anderson said so earlier, but it was a student, not Mr. Anderson, who said it earlier. However, in the earlier exchange Mr. Anderson affirmed it, which was enough for Kenny to determine it was because Mr. Anderson said so. Mr. Anderson regularly affirmed students' answers and the purpose appeared to be to reassure them that their thinking was correct, however, their interpretation appeared to be that they could determine the validity of mathematics based on what he said. While Mr. Anderson appeared to affirm students' answers to help them feel more comfortable speaking up in class, it appeared to work against students' willingness to provide further explanations.

While students expressed discomfort at giving explanations in front of the whole class and some indicated they did not want to listen to other students' explanations (Student Questionnaires, June 27), students seemed more willing to explain their thinking when doing seatwork and they talked to Mr. Anderson alone or in a small group. In those situations students sometimes gave explanations focused on the concepts, but the purpose was to check their reasoning, not to share ideas with the class. For example, while Carol was solving a rational inequality, Mr. Anderson walked next to her desk and looked at her paper:

> Mr. Anderson: You can't multiply by the,
>
> Carol: I know, because of the inequality, we don't know when it's negative. I wondered when I was doing it. (Observation, July 5)

Carol was quick to give a reason, perhaps to make sure Mr. Anderson knew she knew why. However, she did not usually reveal her ideas during whole-class discussion.

Another sociomathematical norm concerned the use of student errors. In classes with sociomathematical norms focused on students' own sense-making, teachers may ask all students to investigate contradictions (Kazemi & Stipek, 2001) or use errors as springboards into deeper inquiry (Borasi, 1994), providing opportunities for students to continue thinking about the concepts. However, in this class, a sociomathematical norm related to procedural errors was that Mr. Anderson would immediately determine whether a student's answer was right or wrong:

> Mr. Anderson: Okay, this is going to be *u* right here, but what's this? [slight pause, student voice can be heard] I mean, we're hoping it's going to be *u* squared. Is that true? If I were to square *y* to the one-third, is that *y* to the two-thirds?
>
> S: Yes.
>
> Mr. Anderson: Yeah, because power to a power,
>
> S: Add exponents.
>
> Mr. Anderson: Close, same base, you add them, power to a power,
>
> S: Multiply.

Mr. Anderson: Multiply. So one-third times two is two-thirds. The 'add' comes like if you had x to the tenth times x to the tenth, then you add them. Power to a power, multiply. So we got a review of exponents there. So this really is u squared like we wanted it to be. (Observation, June 27)

As Mr. Anderson indicated, the rules of exponents were review; students should have gained a conceptual understanding of the rules when they were introduced at least two quarters before precalculus. In general, when students gave a wrong answer for a procedure, Mr. Anderson determined the correctness of students' answers and then either told them the rule or explained the concept. However, most answers offered during whole-class discussion were correct while it was clear many students were confused about the concepts and procedures during seatwork, indicating that students usually only answered during whole-class discussion when they thought they were correct. Thus, many students were not willing to expose their misconceptions during whole-class discussions.

In the following episode, although the first student response was wrong, Mr. Anderson waited for more students to answer (Fieldnotes, July 7). They discussed the piecewise function

$$f(x) = \begin{cases} |x|, & \text{if } -3 < x < 3 \\ -2x + 9, & \text{if } x \geq 3 \end{cases}:$$

Mr. Anderson: All right, what about the domain here? [pause 5 seconds] What x-values can we use? [pause 19 seconds].

S: It'd be all real numbers.

Mr. Anderson: So we got one vote for all real numbers. Anyone else, second that vote?

Sarah: Um, well all numbers bigger than negative three.

S: Positives.

Mr. Anderson: Positives? So we got positives, bigger than negative three, and all real numbers [continues with uni-directional communication to test a variety of input values inferred by the given domains]. (Observation, July 7)

Mr. Anderson did not immediately refute the first answer, nor did he ask the student how he determined the answer (a similar instance noted on June 30). If the first answer offered had been correct, Mr. Anderson may not have offered the opportunity for students to continue thinking. However, his wait time of 24 seconds gave students an opportunity to think and offer ideas, but students resisted making themselves vulnerable and waited for the teacher to explain or ask questions to help them think through problems during the whole-class discussion. Pauses allowed opportunities for them to justify their answers, but they did not use them that way. Eventually, though, Mr. Anderson asked closed questions to guide students to the correct answer. As discussed earlier, when students appeared to be confused about a concept, Mr. Anderson refuted and told as was the norm, but then introduced examples or seatwork that provided an opportunity for students to use the concept. Thus, the sociomathematical norms related to student errors were affected by the social norm that Mr. Anderson would eventually provide the correct answer and an explanation.

Mr. Anderson’s beliefs about mathematics were different from his students’ beliefs. On their questionnaires, eleven of thirteen students thought memorizing steps and formulas was important (see Table 27). However, when Mr. Anderson discussed his beliefs about mathematics, he still discussed the procedures, but liked the flexibility,

Every problem that's out there, you got ten different approaches you could take. As long as, you have these few rules, and students sometimes take them out of proportion and think that all these things are rules, but it's really just built on these core rules and then as long as you stay within that framework, you know you can take any approach you want, and I like that. (Mr. Anderson Interview, July 5)

Mr. Anderson liked the flexibility and creativity offered by mathematics but realized that many students believed that mathematics was about following rules.

Mr. Anderson modeled his thinking. As discussed in previous sections, another sociomathematical norm was that Mr. Anderson interpreted and did the mathematics. For example, Carol asked him to do a homework problem of finding the velocity a fish must swim up a river that flows at 5 mph to minimize their energy output, given by the function, $E(v) = 2.73v^3 \frac{10}{v-5}$ (Stewart, et al., 2002, p. 205).

Carol: I don't think I put it in the calculator right.

Mr. Anderson: [He typed it in his calculator.] Hm, that's interesting [pause of 25 seconds while he walks over to his text and rereads the problem]. Oh, okay, I think partly here, what we gotta do is understand what's going on with the situation. So, when I did my graph I get this picture that looks like it's going like this, and then it drops off, then it looks like it shoots back up.

Carol: Mhm.

Mr. Anderson: So, when I did my graph [he draws it on the board.] in a standard window I get this; if I got it graphed right (see Figure 5).

Figure 5. Graph of a rational function.

Carol: That's what I got on my graph too; that's why I thought I did something wrong.

Mr. Anderson: Okay, so, well, so first off, maybe we need to find out what's happening in here or maybe we need to find out what's happening to the right. [pointing to graph]. But now let's go back to the problem situation. So,

Daniel: You find the minimum.

Mr. Anderson: Yeah, actually this is going to go down to negative infinity; there's no minimum there.

Carol: Yeah.

Mr. Anderson: But, let's think about what's happening. The fish has gotta swim; it's trying to go upriver.

Carol: Right.

Mr. Anderson: Now the current is five miles per hour that way.

Carol: Okay.

Mr. Anderson: So to actually go upriver, what's the minimum speed the fish can go?

Daniel: Five point one.

Carol: Five point one.

Mr. Anderson: Five point, uh one, yeah, five point something. In other words, none of these values that you see in here matter [He crosses out the graph to the left of $x = 5$].

Carol: So greater than five, okay.

Mr. Anderson: Cuz the fish has to be going more than five miles per hour to make any difference. In other words, we need to find out what's happening to the graph to the right of five. So what I did now is, I went to my table and I'm just arrowing down, six is five thousand eight hundred, seven is four thousand six hundred. So we need to be up in the five thousand range on my *y*-values. So on my window I'm going to change my *y*-values to, maximum, maybe seven thousand or something. Yeah, and you can see the little, now ignoring this, the new graph shows up, again, I don't care what happens to the left of *x* equals five, [draws a new graph] but I have a graph that's doing this now, and we need that point [pause 8 seconds]. I just left my *x*'s negative ten to ten.

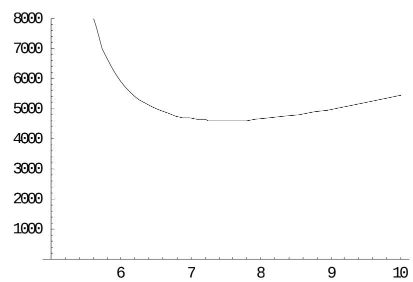

Figure 6. Graph of a rational function, new window.

Daniel: What did you put your *y* at?

Mr. Anderson: I put my *y*-max up at seven thousand.

Daniel: What did you do your min at?

Mr. Anderson: Oh, it doesn't matter, zero; I left it at negative ten. So, a couple things happened there; the first picture you get [inaudible] what you do with that. So the second thing, on this one, since it was a story problem you had to return to the given information to figure out that we really needed *x*-values greater than or equal to five, actually greater than five. So we really needed the *x*-values to the right of five…[they finish the problem]. (Observation, July 18)

Mr. Anderson modeled his thinking and modeled sense-making throughout the preceding episode and did not appeal to rules or memorized procedures as mathematical authorities. However, since he worked the problem, students may conclude from repeated similar responses that teachers must be intermediate authorities for making sense of mathematics (Smith, 1996). Since it was a homework problem, students had the opportunity to think about it and try different strategies to solve it. However, some students may not have continued trying to solve problems when they encountered difficulties since they knew Mr. Anderson would do the problem if they asked about it. Thus, this sociomathematical norm may have helped maintain students' absolute knowing.

The impact of the graphing calculator. The use of graphing calculators was a salient feature of this classroom and influenced students' ways of doing mathematics and ways of interacting. The standards for mathematics in two-year colleges, *Crossroads in mathematics: Standards for introductory college mathematics before calculus* (AMATYC, 1995) espoused the use of technology as "an essential part of an up-to-date curriculum" (p. 2). In this class, sociomathematical norms related to the use of graphing calculators included checking their thinking by using multiple representations and exploring ideas.

Mr. Anderson required the use of a graphing calculator and did not limit its use. Mr. Anderson spent thirty minutes on the fourth day of class using examples to familiarize students with its features and keystrokes; only Kathy did not have one yet, but she moved closer to look on Anthony's. Throughout the quarter, students used calculators to examine multiple representations of functions and both local and global characteristics of graphs (extrema, points, intercepts, increasing/decreasing, general shape, end behavior, and asymptotes). In addition, students used graphing calculators to help them factor polynomials. Mr. Anderson told the class the calculator provided the same information they could obtain by hand, but the calculator did it faster, and the table and graph features should be used in combination to sketch a better graph; it was a tool, but they still had to think (Fieldnotes, June 30). He also suggested they use the table feature to determine good window parameters. Later, while graphing $y = 5x^3 - 40x^2 + 15x - 20$ during seatwork:

> Student to Carol: What did you use for your window?
>
> Carol: I have my window at *y*-minimum, negative three hundred, *y*-maximum, five hundred and sixty.
>
> S: Sheesh.
>
> Daniel to Carol: Negative three hundred and what?
>
> Carol: But I didn't go quite low enough. Go to your table and take a look at the values for *y*.
>
> Daniel: Oh, yeah. (Observation, June 30)

Carol repeated Mr. Anderson's suggestion to use the table to determine appropriate values for the viewing rectangle. Later, during other lessons, Daniel suggested using the table feature to verify points on the graph. For example, the class was determining intervals of increase and decrease for $f(x) = -|x-3| + 2$ and a student answered the interval of increase was $(-\infty, 3]$. Looking at the graph on her calculator, Carol asked, "And three's okay even though it's not exactly three?" to which Daniel responded that the table verified that it was exactly three (Observation, July 11). Daniel found that the calculator could verify his thinking and had already checked to verify the point as belonging to the function when Carol asked her question. In fact, Daniel used the calculator to check rather than ask Mr. Anderson. This is very similar to the sociomathematical norm found by Hershkowitz and Schwartz (1999) where students used technology to refute or confirm their ideas.

Mr. Anderson also used the graphing calculator to have students explore. Assigned exercises from the text provided some explorations which students asked about in class (Fieldnotes, July 11), but in another instance Mr. Anderson asked students to explore and determine how each coefficient in a quadratic function affected the graph (Observation, July 13). Mr. Anderson valued the use of the graphing calculator to explore and gave students the opportunity although he indicated it was something extra and not part of the expectations of the course.

Some students used the calculator to explore independent of Mr. Anderson. For example, during a seatwork assignment students worked to find a function of the form $n(t) = n_0 e^{rt}$ that models the population of a certain rare species of bird over time; the given information included a graph containing the points (5, 3200) and (0, 1500) (Stewart et al., 2002, p. 392). Daniel, Carol, and Sarah discussed differences in their answers and decided the differences could be attributed to the number of decimal places they used for the value used for *r*. Daniel mentioned that when he used three decimal places instead of four, the answers differed by 80. However when he used seven decimal places instead of four, the answer differed by only three (Fieldnotes, August 9). This conversation took place independent of the teacher.

Sarah and I discussed graphing calculators and she admitted she had not wanted to learn to use it because there were so many buttons, but after Mr. Anderson used a class period to help them learn to use it, she changed her mind. When I asked if she used it to explore, she replied, “Sometimes when I'm doing equations or whatever in homework, I'll either switch the, like a negative to a positive, and just kind of see what that does, and that's fun” (Sarah Interview, July

13). So, incorporating the graphing calculator into the course provided opportunities for individuals to explore when they were curious and fostered communication between students as they discussed their explorations.

Summary of Research Question Two: Interactions Related to Mathematics

Communication levels remained at uni-directional and contributive, since roles and social norms affected the possibilities for more complex communication. While social norms provided students many opportunities to ask questions and make comments, the nature of their input indicated they were absolute knowers and expected Mr. Anderson to be mathematical and intermediate authority. His responses did not contradict their expectations.

Interactions in this class as they related to mathematical activity focused on concepts and procedures. However, while reasons for new procedures were provided, students were rarely expected to be able to explain those reasons. While students asked many questions and made comments throughout each lesson, Mr. Anderson used their questions and comments as opportunities to explain clearly and to devise new examples and seatwork to clear up their confusions.

The sociomathematical norms of this class related to explanations and error emphasized intermediate authority with Mr. Anderson, while sociomathematical norms related to the use of graphing calculators encouraged students to explore and to verify their own ideas and encouraged the use of multiple representations. In addition, sociomathematical norms provided that Mr. Anderson did the mathematics while modeling his reasoning and sense-making. Roles and social norms affected the nature of the sociomathematical norms that developed so the evaluation of mathematical activity centered on whether Mr. Anderson validated or refuted it. Also, students' questionnaire responses indicated they valued Mr. Anderson's explanations over their own or their peers, and so resisted giving thorough explanations during whole-class discussions. During my observations of whole-class discussions, five students did not speak up to answer even short-answer questions more than twice unless they were called upon. Pressing a student in front of the whole class seemed to violate Mr. Anderson's goal of making the classroom environment comfortable. While Yackel and Rasmussen (2002) found that an instructor was able to initiate and sustain sociomathematical norms related to explanations in a college differential equations class, they emphasized that they discussed and then supported the formation of these roles with students. They did not say how many of their students participated in the way they were expected, although they demonstrated that some did. In the present study, Mr. Anderson did not discuss classroom norms and roles with students.

Chapter Six: Discussion and Implications

As described in chapter one, this study was designed to investigate students' ways of knowing in light of the development of community and interactions in their mathematics classrooms. I used a qualitative two-case study design to investigate the development and nature of classroom community and the nature of interactions related to mathematics in two community college precalculus classes.

In this chapter, I summarize the results for each of the two research questions, then discuss the relationships between community, interactions related to mathematics, students' ways of knowing, and student learning. In particular, I offer a refinement of Boaler and Greeno's (2000) dichotomy of didactic versus discussion-based classes by examining how specific factors related to community and interactions affected students' ways of knowing and opportunities to learn. Finally, the last sections provide implications for practice and further research.

The Development and Nature of Community

The first research question addressed the development and nature of community. In this section, I describe how the roles and social norms of the two classes developed and compare them.

Students' responses to the questionnaire given on the first day of class indicated their expectations for their roles and the instructors' roles. Students in both classes started the term with similar expectations for their roles: they expected to practice, follow examples, and listen to the instructor explain. This description echoes the descriptions of didactic classes provided by students interviewed by Boaler and Greeno (2000). However, more students in the present study also responded that they wanted to work with their peers than wanted to work alone, and most students indicated they wanted instructors who involved the class in discussion, descriptions similar to discussion-based classes (Boaler & Greeno, 2000). However, when asked specifically about their roles in discussion, students perceived their roles to ask questions and answer questions if they could offer the correct answer. Some responded that they would rather not participate in discussions although they wanted to listen to them. Consequently, students' roles and classroom social norms were initially constrained by the conceptions they held of their roles and the teacher's role.

Students' roles differed in some ways in the two classes. Mr. Anderson's class was a discussion-based classroom community since he provided many opportunities for students to work together and discuss the mathematics with him in whole-class discussions (Boaler & Greeno, 2000). Because of this affordance, students asked many more questions in Mr. Anderson's class than in Mr. Reilly's class. Mr. Reilly did not allow group or seat work, but provided opportunities for students to interact with him throughout lecture, although with fewer opportunities to contribute than in Mr. Anderson's class. Some students in Mr. Anderson's class asked for specific examples and Mr. Anderson used these suggestions, student questions, and student work he observed during seatwork to inform the direction of instruction. In fact, students came to expect to be able to ask questions and that Mr. Anderson would slow down, provide new examples, and explain until they understood. Karabenick and Sharma (1994) found that providing opportunities for students to ask questions and providing high-quality answers such as detailed explanations, affected motivation and encouraged students to formulate questions. In Mr. Anderson's class, students' formulation of questions demonstrated a higher level of cognitive demand than simply asking for homework solutions.

The opportunities Mr. Anderson provided for students to engage in group work also created a relational classroom community and increased the likelihood students would engage in class discussions and discuss the mathematics with each other. Mr. Anderson's classroom environment appeared to have encouraged two students, Kathy and Kevin, to change their perceptions of their roles and answer questions during whole-class discussions. In fact, participation in seatwork with others seemed to be the determining factor in whether students participated in whole-class discussions since those who worked with others during seatwork were more likely to speak up during whole-class discussion even if they responded on their questionnaires that they would not speak up in whole-class discussions. Opportunities to work together also influenced some students who initially responded that they preferred to work alone during class, such as Daniel, Kevin, and Kenny, to work with others.

Yet, student roles in the two classes during whole-class discussion were more alike than different. In both classes, students gave short answers and almost no explanations, and most students usually only answered if they thought they knew the right answer. Student roles did not include sharing ideas with other students or listening to other students. The instructors supported these roles since they made very few attempts to encourage students to explain or to influence the nature of student explanations. They answered all questions with thorough explanations rather than shift the responsibility for thinking back to the students (van Zee & Minstrell, 1997). In addition, the instructors also interpreted and expanded on students' short answers rather than ask the students what they meant.

Student questionnaire responses indicated they had very little experience sharing their ideas or listening to other students share ideas. While a few students responded that they wanted to hear other students' ideas, several specifically replied they did not want to hear other students' explanations. This indicated they may have been in classes where students explained or shared solutions but did not yet know how to listen critically and learn from their peers. However, providing opportunities for students to interact in whole-class discussion encouraged more interaction as the term progressed, so students became more comfortable participating in these classes, which allowed the classes to increasingly become relational.

Instructor roles also contributed to the development of a relational community. Both instructors' roles resonated with support and approachability, although Mr. Reilly made himself available to students outside of class while Mr. Anderson made himself available to students during class. Mr. Anderson repeatedly demonstrated helpfulness throughout class by establishing a comfortable classroom, providing extended wait-times and many opportunities for students to ask questions and receive feedback. He connected with students by walking around to talk to individuals and using students' names. Mr. Reilly also graciously entertained questions although he did not provide as much time or as many opportunities for them. He demonstrated his support by empathizing with students and providing access to the mathematics through his use of informal language. He waited after class to spend time conversing with students who wanted to get to know him.

While students perceived their instructors' willingness to answer questions as supportive, Mr. Reilly and Mr. Anderson accepted all responsibility for responding to questions. As a result, whole-class discussions remained between instructors and students, rather than including student-to-student communication. Instructors approved, refuted, and expanded on students' answers, maintaining their mathematical and intermediate authority. This instructor role constrained students' formation of their own ideas and precluded the need to listen to their peers' ideas. While building community is important, it was also important to consider the nature of interactions related to mathematics.

The Nature of Interactions Related to Mathematics

Because the social norms and roles sustained intermediate authority with the instructors, communication in both classes remained unidirectional and contributive. This idea, posited by Brendefur and Frykholm (2000) was clearly shown in the present cases. Unidirectional communication in both classes included many closed questions and some lecturing. Contributive communication did not require students to listen to each other and students rarely responded to each others' ideas during whole-class discussions.

However, both of these communication types allowed the instructors to determine some students' current conceptions of the mathematics and adjust their instruction, a component of instructive communication. Instructive communication includes communication through which instructors understand students' current conceptions and alter their instruction to modify students' conceptions. Mr. Anderson regularly adjusted instruction based on questions, answers, or comments from students, often introducing new examples to clarify the conceptions, while Mr. Reilly only adjusted when none of his students indicated understanding of mathematical identities. I did not consider these episodes in Mr. Anderson's class instructive communication though, because students continued to rely on instructor explanations and did not necessarily modify their current understanding but accepted explanations without reconsidering their old conceptions.

In spite of the lower-levels of communication, Mr. Reilly emphasized the importance of learning concepts and rarely demonstrated procedures. Boaler and Greeno's (2000) description of didactic classes suggested that in classes that were individualistic, the mathematics necessarily focused on watching teachers demonstrate procedures which students later practiced alone, and that mathematics was dominated by rules to memorize. Mr. Reilly's case is an example of an individualistic class that focused on conceptual understanding. Both instructors consistently used multiple representations when explaining concepts, encouraged exploring, and discussed and modeled sense-making.

Several tasks assigned by the instructors in these classes could be characterized as “doing mathematics” or procedures with connections (Henningsen & Stein, 1997). Mr. Reilly presented problems and suggested students needed to work outside of class to solve them, providing no time in class for students to work individually or with others to solve problems. While Mr. Anderson expected students to attempt homework problems when he had not discussed the procedures, believing students should make conceptual connections, he worked the problems in class when students indicated they could not do them. For the tasks he assigned in class, the social norms, roles, and students’ ways of knowing affected the implementation of the tasks so they were not implemented at the high level of cognitive demand intended. The interactions described above show that while the instructors valued students’ implementation of tasks at a high level, they either did not provide opportunities, or did not know how to support implementation at high levels.

There was evidence that some students did not know what to do if they were not shown how to do problems in class. Sarah, in Mr. Anderson’s class, and Steve, in Mr. Reilly’s class, reported they did not know how to learn the mathematics when they were out of class and did not understand. Other students had goals of just passing their class and may not have been willing to spend the time and effort necessary outside of class to solve problems. This implies that providing students opportunities to work on tasks during class and supporting their implementation at the level of cognitive demand intended may be necessary to help students develop skills and dispositions crucial to developing as mathematics learners.

In general, a sociomathematical norm during whole-class discussion in both classes was that the instructors did the mathematics. The instructors stayed at the boards, drawing multiple representations and explaining their thinking and pointing out connections to help students follow their explanations. Mr. Anderson sometimes asked questions and followed students' responses to perform procedures, but he only followed correct suggestions. This sociomathematical norm reinforced students' ideas that their instructors' roles were to interpret and show them the mathematics.

Two sociomathematical norms observed in these classes may have contradicted students' ways of knowing. Mr. Reilly's insistence that students must have reasons for mathematical procedures and his explicit appeals to understand rather than memorize appeared to influence their perceptions of doing mathematics. In Mr. Anderson's class, the use of graphing calculators supported students' willingness to explore on their own and discuss differences in their solutions with each other. Both of these sociomathematical norms allowed students to think in context about mathematical ideas, and share their own ideas, which supported more complex ways of knowing (Baxter Magolda, 1992).

Discussion

In this section, I discuss how the development of community, students' ways of knowing, and the nature of interactions related to mathematics were interrelated and affected students' construction of mathematical concepts.

The Nature of Community Affected Interactions Related to Mathematics

The roles and social norms that developed in these classes affected the nature of interactions related to mathematics. Since the social norms and roles fostered intermediate authority with the instructors, they constrained opportunities for students to engage as members of a mathematical community who engage in the types of mathematical discourse envisioned by current reformers (AMATYC, 1995; NCTM, 1991, 2000). Cobb and Yackel (1995) describe the relationship between the development of roles and social norms and students' beliefs about the roles and social norms as reflexive. The students in their study reorganized their beliefs about their roles and classroom social norms as they renegotiated the roles and social norms initiated by their teacher. However, in the present study, since traditional roles and social norms were not challenged, students were able to maintain their beliefs about the instructors' roles as authorities and their roles as recipients of knowledge.

Boaler and Greeno (2000) suggested that students in discussion-based classes became relational agents, helping each other understand mathematical concepts. In Mr. Anderson's class, students were able to work together but still asked Mr. Anderson if they were correct, and their questions of him were more procedural than conceptual. While I could not hear much of the conversations between students during seatwork, based on the types of questions they asked in class, it is likely that their discussions focused on using procedures correctly rather than deepening their understanding of the concepts. Unless tasks and whole-class communication challenge students' conception of mathematics as procedures and rules used to produce correct answers, fostering a discussion-based class alone is unlikely to change the way they communicate with each other. However, fostering such a community may be necessary to help students feel comfortable enough in class to contribute during whole-class discussions.

The social norms of whole class discussion and instructors' roles as authorities encouraged funneling, guessing, students' pseudo-conceptual behavior, and instructors' acceptance of one correct answer as evidence the class understood. These factors tended to decrease cognitive demand. In addition, the social norms and roles constrained communication to unidirectional and contributive communication, rather than providing opportunities for reflective and instructive communication. The nature of reflective and instructive communication consists of discourse in which students transcend sharing ideas or asking the instructor questions to listening to peers and striving to make sense of the mathematics in their own way (Brendefur & Frykholm, 2000). Sustaining these types of communication relies on classroom social norms that encourage students to listen to each other, and instructor and student roles that encourage students to author their own understanding.

Relationships Between Community and Students' Ways of Knowing

Boaler and Greeno (2000) described discussion-based classes where students worked in groups and discussed mathematics with their teacher and peers. Students in these classes described the relationships they formed as central to their learning. The authors contrasted the affordances of this type of mathematical community with the constraints of didactic classrooms where the teacher demonstrated procedures which students were expected to practice alone. They concluded that the didactic community afforded only received knowing while the discussion-based community supported more complex ways of knowing.

In the present study, Mr. Anderson fostered a discussion-based community by providing ample opportunities for students to suggest ideas, ask questions, and work together. However, some students in Mr. Anderson's class did not interact with peers, and during seatwork, students still appealed to his authority to determine the correctness of the mathematics instead of their own sense-making. These stances indicated their absolute ways of knowing. The characteristics of this case indicate that creating a relational community does not necessarily challenge students' current ways of knowing. However, these roles and social norms confirm students' ways of knowing. Baxter Magolda (1992) found that absolute and transitional knowers appreciated instructors who entertained questions and provided ways for students to be active during class rather than solely listen to lecture, and this level of activity was a way to confirm students' ways of knowing. Confirming students' ways of knowing was important in "heightening students' interest in learning, strengthening their investment in that process, creating comfortable learning atmospheres, and developing relationships that foster understanding" (p. 268).

However, teachers must balance confirmation with contradiction of students' ways of knowing to foster more complex ways of knowing (Baxter Magolda, 1992). Discussion-based communities are unlikely to support students' growth in more complex ways of knowing unless teachers suspend their authority and challenge students' current ways of knowing by challenging their roles as receivers of knowledge: "Students do not view themselves as knowers until the learning environment implies or states directly that they have something of value to say... Failure to validate the student reinforces absolute and transitional ways of knowing" (Baxter Magolda, p. 273).

Other factors of community that emerged in this study may also support growth in ways of knowing. Teacher roles of support and offering students opportunities to know them are important in fostering more complex ways of knowing (Baxter Magolda, 1992). In both classes, students I interviewed valued their instructors' approachability and support in helping them understand. Baxter Magolda (1992) indicated these as factors that supported and appropriately confirmed students who were absolute and transitional knowers. Appropriate confirmation "sets the stage for students to participate in learning and to become creators of knowledge rather than recipients of it" (p. 269).

Similar to freshmen in Baxter Magolda's (1992) study, the majority of students in both classes started the term as absolute knowers, believing mathematics knowledge was certain and the instructor had this knowledge and could impart it to them. However, some students were not freshmen, such as Carol and Sarah who both had bachelor's degrees, and Tim, who graduated after completing this class, yet they were still absolute knowers in their mathematics classrooms. It is likely that the nature of mathematics as they had encountered it as students portrayed mathematical knowledge as certain.

Interactions Related to Mathematics Affected Students' Ways of Knowing.

Interactions related to mathematics included sociomathematical norms, communication about mathematics, the use of technology, and the way the instructors portrayed mathematics. As discussed earlier, the social norms and roles constrained sociomathematical norms and communication that entail students' justifying and explaining, sharing their own ideas, and making sense of mathematics by reflecting on their own and their peers' mathematical arguments. Interaction of this nature evidences contextual knowing. Since community did not foster this type of interaction and Baxter Magolda (1992) found only 2% of college seniors to be contextual knowers, it is not surprising students in this study did not interact at this level.

Some factors of interactions may have supported growth in more complex ways of knowing by situating learning in students' experiences (Baxter Magolda, 1992). Mr. Anderson used student questions and comments to guide his instruction and provided opportunities for students to work problems in class. By connecting to students' conceptions of the mathematics as they experienced it, he situated learning in their experiences.

Mr. Reilly situated learning in students' experiences in different ways. He incorporated stories and real life applications into the lessons to motivate and connect the mathematics he was teaching to their experiences (Baxter Magolda, 1992). The students I interviewed clearly valued these elements and suggested it helped them connect to the mathematics. In addition, Mr. Reilly specifically mentioned that his perspective of the mathematics differed from the authors' of the textbook suggesting that mathematical knowledge is uncertain. Baxter Magolda recommended that instructors need to provide opportunities for students to increasingly see knowledge as uncertain: "Clarifying that the information presented comes from a particular perspective and is generated by other human beings is essential for students to begin to see themselves as capable of forming their own perspectives" (p. 278).

Students' Ways of Knowing Affected Learning

The results of this study suggest that students who have different ways of knowing may have different criteria for making sense. For example, Sarah evidenced absolute knowing throughout the study. When I interviewed her, she described a procedure as making sense after the instructor explained how it connected to a concept they had just learned. Before his explanation, the procedure did not make sense to her because it conflicted with an earlier idea. She did not try to resolve the conflict, but was satisfied that the instructor's explanation supported the new rule. Absolute knowers appreciate understanding but still believe most knowledge is certain, so they accept explanations from instructors without critically questioning them. Likewise, Carol doubted her own sense-making on several occasions and wanted Mr. Anderson to verify her answers.

In contrast, by late in the term, Tim spoke up during whole-class discussion to challenge Mr. Reilly's answers, explaining that the solutions to a trigonometric equation did not make sense because they produced the wrong signs. Although his questionnaire responses indicated he was an absolute knower at the beginning of the term, in this episode Tim showed evidence of contextual knowing. Contextual knowers make sense by thinking through problems, striving to understand concepts in contexts, and comparing ideas (Baxter Magolda, 1992). This implies instructors should discuss and model what it means to make sense.

Implications for Practice

The results of this study suggest classroom environments described by mathematics educators for K-12 students also support more complex ways of knowing of college precalculus students. The NCTM (2000) proposed students make sense by looking for patterns, making connections, and by engaging in discourse where they listen to others and share their ideas. Similarly, contextual knowers learn by engaging with ideas in context, listening to their peers' ideas and sharing their own; they believe the instructor's role is to facilitate discourse that helps them integrate knowledge without occupying a role of mathematical authority (Brew, 2001).

Student and instructor ideas about the nature of mathematical knowledge and how mathematics is learned affect and are affected by community and interactions related to mathematics. Students' expectations and the ways they participate in class evidence their ways of knowing, while instructors may maintain, confirm, or challenge students' ways of knowing by the constraints and affordances they provide. Table 32 provides a framework containing evidence of instructor and student perspectives and roles in several domains similar to those discussed by Baxter Magolda (1992) and additional domains considered in this study. Domains at each level include the nature of mathematical knowledge, the nature of learning mathematics, role of peers/ social norms, relationships, interactions related to mathematics/ communication, and students' ways of knowing.

The framework is similar to one developed by Franke, Carpenter, Levi, and Fennema (2001) which described levels of teachers' engagement with children's thinking. The four levels in the framework in Table 32 correspond to students' ways of knowing; lower levels correspond to absolute and transitional knowing while higher levels correspond to independent and contextual knowing. When instructors and students are both at Level 1, absolute knowing is maintained. However, when instruction is provided at levels higher than students evidence, students' ways of knowing are confirmed and challenged. In the rest of this section, I discuss each domain separately, including ways instructors may increase opportunities to challenge students' ways of knowing, and describe the levels evidenced by Mr. Reilly, Mr. Anderson and their students for each domain.

The *nature of mathematical knowledge* is a subset of Baxter Magolda's (1992) domain of students' beliefs about the nature of knowledge. Students' beliefs about the nature of mathematical knowledge span a continuum of certain to uncertain, and are evidenced by their openness to multiple ways of solving problems, recognition of different perspectives, and their appreciation of real life applications of mathematics. The main difference between a Level 3 student and a Level 4 student is that Level 4 students recognize they have a unique perspective of mathematical ideas.

Instructor portrayal of mathematics can affect student beliefs about the nature of mathematical knowledge (Baxter Magolda, 1992). Instructors portray mathematics as certain when they divorce it from its development, history, and real life, or when they imply there is only one way to solve each problem. In contrast, they can challenge students' belief in the certainty of mathematics by relating stories that help students understand the nature and history of the mathematics they study or by providing opportunities for students to engage in mathematical activities from which they can develop understanding. At the highest level, instructors foster and respect students' individual construction of mathematical ideas.

Mr. Reilly intentionally and explicitly challenged the notion of mathematics as certain. He depicted mathematics as developed by people and useful in real life, discussed that authors and instructors can have different opinions, and suggested there were many solution paths to some problems. However, while sometimes suggesting students needed to find their own ways of thinking about the concepts, he more often implied students needed to adopt his reasoning. For these reasons Mr. Reilly portrayed the nature of mathematical knowledge at Level 3. In their interviews, two of his students, Natalie and Steve, reported that the stories and real life connections made the mathematics far more interesting and motivated them, demonstrating that Mr. Reilly's efforts were enough to challenge their previous conceptions of mathematical knowledge.

Mr. Anderson did not intentionally portray mathematics as developed by people throughout history and rarely connected the content to real life situations, but he supported the idea that there were many solution paths to the same problem and recognized and used student suggestions. However, while Mr. Anderson responded to student questions and mathematical ideas, there was no evidence the students tried to develop their own perspectives, but instead accepted explanations from Mr. Anderson without critical examination. Because he was willing to explain completely in response to their ideas, Mr. Anderson's intermediate authority interfered with students' development of their own perspectives. In order to support contextual knowing, instructors need to support students' reflection on their own suggestions and questions by rejecting the role of authority.

Students' ideas of the nature of mathematical knowledge were not directly addressed in the Student Questionnaires, but in their responses to several questions, many students expressed a need to be told how to do the mathematics and expressed a need to see step-by-step examples of how to complete procedures, evidencing Level 1. However, a couple of students indicated they wanted to see multiple ways of doing problems, and one student responded he wanted to see real life connections to the mathematics, evidencing Level 3.

The second domain concerns the *nature of learning mathematics*, and descriptions include evidence of how students strive to learn mathematics. In general, the levels in this domain distinguish between those who try to learn procedures without understanding the underlying concepts, and those who strive to understand the concepts and develop procedural fluency. The domains span the four levels of cognitive demand described by Henningsen and Stein (1997): memorization, procedures without connections, procedures with connections, and doing mathematics. Distinctions between Levels 2 and 3 involve the extent to which students attempt to use understanding, since Level 3 students strive to understand and use explanations from authorities. Level 4 students gain insight through reflecting on their mathematical activities such as exploring and making and testing conjectures. Rather than accept explanations from experts, they compare them to their current understanding of concepts and reconstruct their understanding.

The instructor column in the *nature of learning mathematics* presents evidence of the instructional strategies instructors employ. Level 1 describes the didactic classes referred to by Boaler and Greeno (2000) in which instructors focused on demonstrating steps and showing students how to successfully reproduce procedures. Higher levels in this domain stress understanding and providing opportunities for students to solve problems and apply their reasoning. In addition, instructors use student thinking to guide their instructional decisions. The difference between Level 3 and Level 4 is Level 4 instructors support students' development of concepts through engagement with rich tasks, reflection on activities, and are able to support students in maintaining the high cognitive demand. It is likely an instructor's own understanding of the concepts and connections influences their ability to raise their level in this domain.

In the present study, Mr. Reilly developed concepts and connections and discussed learning mathematics as conceptual. He presented tasks so students could make connections or engage in problem solving. However, since he did not provide opportunities for this activity during class, he could not respond to student thinking, nor provide scaffolding for reflection and communication. In addition, he sometimes expected students to make sense in a specific way rather than allowing them to construct their own understanding in a way that made sense to them. For these reasons, Mr. Reilly was a Level 3 in this domain. Mr. Anderson explained ideas to help students understand and connected new procedures to concepts. He often expected students to use conceptual understanding when employing procedures, a Level 3 indicator, but on occasion, when understanding was difficult, he allowed students to memorize, evidence of a Level 2. He also used student thinking to guide instruction, a Level 4 characteristic. He most often evidenced a Level 2 in this domain.

Most students in both classes evidenced Levels 1 and 2 on their Student Questionnaires since they indicated they learned best by listening, being given step-by-step procedures, and following examples. Level 2 students want to hear explanations from their instructor and responses on the questionnaire indicated this was important to most students. Observational evidence indicate students in Mr. Reilly's class remained at Levels 1 and 2, listening to questions and answers, and listening to Mr. Reilly's explanations, although it was clear several times they did not understand some explanations. An exception to Level 1 and 2 evidence was when Tim pointed out what he thought was a mistake in Mr. Reilly's solution (Appendix H). During this episode, Mr. Reilly solved a problem but Tim argued that the solutions did not make sense because they contradicted his understanding, evidence of Level 4.

The third domain, the *role of peers and social norms*, focuses on the value of student ideas to support and challenge their peers' concept development and understanding. Students at the lowest level do not want to listen to peers because it may confuse them, while students at the highest level are willing to listen to their peers' ideas and offer their own ideas for others to consider. At the highest level, students engage in mathematical argumentation with each other and determine correctness of the mathematics by the validity of the argument. For instructors, the domain encompasses their expectations of and ability to support students' engagement in mathematical arguments. At Level 4, instructors realize they must reject roles of mathematical and intermediate authority so that students listen to their own and peers' ideas and use reasoning rather than reliance on an authority to determine what makes sense. Higher levels are evidenced by sociomathematical norms like those found in Table 2.

Absolute and transitional knowers need support in learning to listen critically to their peers and develop their own ideas. Instructors may provide this support by focusing during whole class discussions to help students listen to each other (Wood, 1998), and by rejecting the role of authority. In order to remove themselves from positions of authority, instructors can remove themselves physically from the center of discussions by requiring student presentations of solutions to the whole class and allowing presenters to field questions. In addition, group work allows more students to discuss and share ideas in a safer size group.

In the present study, very few students responded on their questionnaires that they were willing to share ideas with the class. Of those who said they would, one reason they were willing to share was to show they knew the right answer. In fact, most students only answered questions when they thought they knew the right answer. This behavior may have been encouraged by the implicit message students received when their instructors evaluated responses as right or wrong. Also, once instructors heard a right answer, they explained and moved on as if all students understood. Such teacher responses indicate instructors wanted right answers and that it was important to continue progressing through the material.

Mr. Reilly evidenced Levels 1 and 2 in this domain most of the term since he stayed at the center of class discussions and did not allow group work, but occasionally evidenced a Level 3 when he used student ideas to solve problems. Mr. Anderson primarily evidenced Level 3 since he provided opportunities for students to work together and used their ideas to inform his instruction but maintained intermediate authority. He attempted to have students listen to each other during whole class discussion on at least two occasions.

Student questionnaire responses evidenced mostly Levels 2 and 3 in this domain since about two-thirds of students wanted to work with others, some to check answers and some to share ideas. In Mr. Anderson's class, some students evidenced Level 3 during seatwork, but at this level, students still appeal to instructors' final pronouncements of correctness. In this domain, since instructor levels were at or below student levels, the instructors did not challenge students' ways of knowing.

The fourth domain concerns *relationships* between learners and between learners and instructors. Evidence for students range from not wanting to know other students or the teacher, or wanting to know others to make the classroom more comfortable, to valuing relationships with peers because of their ability to challenge ideas and promote understanding. Evidence for instructors in this domain extend from establishing an environment that stifles relationships to fostering an environment of collaborative learning. Higher levels increasingly include instructor support, opportunities to know the instructor, and opportunities for students to work together.

Boaler and Greeno (2000) reported that students emphasized the importance of relationships when they described their discussion-based classes. The authors also suggested that when classroom communities did not foster relationships, students were constrained to received knowing. The present study supports the related idea that relational communities confirm students' ways of knowing. However, in order to foster more complex ways of knowing, instructors must both confirm and challenge students' ways of knowing (Baxter Magolda, 1992). The current research showed that instructors may maintain mathematical and intermediate authority in relational communities, demonstrating that fostering relationships is necessary but not sufficient to advance students' ways of knowing. In order to advance students' ways of knowing, instructors need to reject the role of authority and promote student reflection on their own and their peers' ideas.

Both instructors allowed students opportunities to know them and offered support at a Level 3, which students valued. However, Mr. Reilly demonstrated Level 1 characteristics because he did not allow students opportunities to work together in class. While Mr. Anderson provided plenty of opportunities for students to work together, he also allowed students to avoid group work by maintaining intermediate authority, evidencing Level 2. Students appreciated Mr. Reilly's efforts to be available and their informal conversations with him. The idea that students valued this characteristic of their instructor was one of the more surprising findings for me, and one I have incorporated into my own teaching. Instructors may increase their levels in this domain by intentionally letting students know them and providing the types of tasks that require student collaboration.

Most students in both classes responded they wanted to get to know others, although a few wrote they only wanted to get to know the teacher. In both classes I observed there were silent students who did not work with others, who arrived as class started and left as soon as class ended, talking to no one. The two quietest students in Mr. Anderson's class responded on their questionnaires they would work with others, however, they did not because they were physically isolated from others and Mr. Anderson answered their questions as he walked around.

The final domain contained in the framework describes student and instructor evidence concerning *interactions related to mathematics/ communication.* This domain focuses on mathematical communication as described by Brendefur and Frykholm (2000) and the use of tools such as technology. Student evidence at the lowest level includes giving only short answers that are cue-based, indicate pseudo-conceptual behavior or memorized procedures. Students at this level ask questions seeking a rule and may use drill software to practice steps and follow examples. Student evidence at Level 2 includes joining in contributive communication but not to help their peers learn, so when forming their explanations they may not consider the way other students may interpret it. Students ask questions to elicit explanations from their instructor and use technology to illustrate multiple representations although they may not reflect on connections. Level 3 evidence includes students discussing their mathematical ideas more than at lower levels and using technology to explore. Finally, Level 4 students participate in reflective and instructive communication; if technology is used, it may be used to make and test conjectures independent of the instructor. Instructor evidence in this domain ranges from fostering only unidirectional communication to fostering reflective and instructive communication. Use of technology ranges from drill software to using technology to explore, support conceptual understanding and sense-making, and make connections between multiple representations.

Absolute knowers expect their instructors to provide clear explanations (Baxter Magolda, 1992), so Level 1 and 2 students may become frustrated when instructors withhold them. However, Lobato et al. (2005) demonstrated that some explanations could evoke sense-making. Rather than provide a procedure when students asked for an explanation, instructors asked questions to determine students' current understanding and then explained important concepts or connections students did not understand. Instructors followed the explanation with a challenge to students to use the new information to solve problems, although the authors found that often students were not able to use the ideas immediately. Nevertheless, in response to students' request for explanations, rather than provide students with efficient ways to find correct answers without understanding, instructors may still satisfy this request and support students' construction of understanding by initiating and eliciting (Lobato et al.).

Mr. Reilly primarily evidenced Level 1 in this domain, although he sometimes asked students how they solved problems and their reasoning for some procedures, a Level 2 characteristic. His students evidenced Level 1 most often, but evidenced Level 2 when they asked questions seeking explanations or explained their own reasoning. Mr. Anderson often attempted to foster interaction at Level 3 since he elicited student contributions and offered new problems based on his understanding of students' current constructions, an element of instructive communication. He also required the use of a graphing calculator and suggested students use it to explore, which they did. Instructors may increase the levels of communication in their class by phrasing their questions carefully to elicit ideas and explanations, and responding carefully so that other students consider their peers' ideas rather than the instructor's response to the ideas.

Evidence at each level provides insight into students' ways of knowing and evidence of the constraints and affordances provided by instructors. When students and instructors are at the same levels, students' ways of knowing are maintained, while instructors who evidence higher levels than their students provide challenges to students' ways of knowing. When instructors evidence lower levels than their students, students may become alienated by the lack of opportunities to develop and reflect on their own perspectives; Boaler and Greeno (2000) found that students who have more complex ways of knowing rejected math because the didactic classes did not provide opportunities to think.

In the introduction, I described students who did not like being asked to think about concepts and share their ideas, and believed their teachers should explain in response to student questions. I described others who, when given opportunities to work together, chose to work alone. It is likely these students were absolute knowers who believed the instructor had the knowledge and should impart it when asked, and that their peers could not help them learn. Instructors attempting to interact at a Level 4 met with resistance. Instructors who want to foster Level 4 students will need to be aware of students' conceptions in each domain and balance confirmation with challenges, and explicitly discuss their expectations and ways students may be successful at these new ways of learning.

Implications for Further Research

Studies connecting students' ways of knowing in college mathematics classes to their learning are necessary; research should be conducted to determine if Level 4 classes lead to more powerful constructions of the mathematics than lower level classes. In addition, research on difficulties instructors encounter as they attempt to establish a Level 4 class is necessary. Specifically, studies similar to the present one, but in which instructors attempt to foster student engagement at a Level 4 could lend insight into ways students respond to expectations of their new roles and ways instructors may support student learning in new ways. The studies should include classes of college students at varying levels of college classes.

Other areas of research could include expanding to more domains. Specifically, assessment is an area that affects students' engagement and study strategies. Research could discern types and levels of assessment that maintain, confirm, or challenge students' ways of knowing.

Further research could also delineate the roles of technology in promoting more complex ways of knowing. In addition to the use of graphing calculators and computer algebra systems to explore and solve problems, students' use of software that allows explorations such as spreadsheets and geometry software may lend insight and challenges to their current ways of knowing. Document cameras and SmartBoards© may also be used to challenge absolute and transitional ways of knowing as students use them for presentations and instructors move away from the center of discussions.

Previous researchers examining ways of knowing considered gender and noticed gender differences (Baxter Magolda, 1992; Benlenky et al. 1986/1997; Brew, 2001). Aware of this, I considered gender differences while collecting data. However, some students of both genders exhibited the ways of knowing of the other gender. For example, Greg exhibited signs of silence and received knowing while Carol, Sarah, and Natalie evidenced mastery knowing. Daniel and Kenny initially did not want to work with others, a male pattern, but discussed mathematics with Carol and Sarah. Because of the anomalies, I chose not to systematically analyze the data for gender differences in ways of knowing. However, this could be pursued closer in a study that analyzes each gender's response to a Level 4 instructor.

Also, the students in the present study indicated absolute and transitional knowing regardless of their year in school, probably because of the way mathematical knowledge is presented as certain in so many mathematics classes. This idea suggests a study to examine how students who have more complex ways of knowing in other contexts adjust to opportunities for independent and contextual knowing in mathematics class.

References

Abramovich, S. (2005). How to 'check the result'? *International Journal of Mathematical Education in Science and Technology 36*(4), 414-423.

Adler, P. A., & Adler, P (1994). Observational techniques. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research. London* (pp. 377-392). London: SAGE Publications.

American Mathematical Association of Two Year Colleges (1995). *Crossroads in mathematics: Standards for introductory college mathematics before calculus.* Memphis, TN: AMATYC.

Angier, C., & Povey, H. (1999). One teacher and a class of school students: Their perception of the culture of their mathematics classroom and its construction. *Educational Review, 51(*2), 147-160.

Baxter Magolda, M. B. (1992). *Knowing and reasoning in college: Gender-related patterns in students' intellectual development*. San Francisco: Jossey-Bass.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986/1997). *Women's ways of knowing: The development of self, voice, and mind*. New York: BasicBooks.

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting.* Philadelphia: Open University Press.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: ABLEX Publishing.

Borasi, R. (1994). Capitalizing on errors as "springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education, 25*(2), 166-208.

Brendefur, J., & Frykholm, J. (2000). Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education, 3*(2), 125-153.

Brew, C. (2001). Women, mathematics and epistemology: An integrated framework. *International Journal of Inclusive Education, 5*(1), 15-32.

Carlson, M., & Oehrtman, M. (2005). Research Sampler: Key aspects of knowing and learning the concept of function. The Mathematical Association of America. Retrieved March 26, 2005, from http://www.maa.org/t_and _l/sampler/rs_9.html.

Cassel, D., & Reynolds, A. (2002). Listening as a vital characteristic of synergistic argumentation for enhanced mathematical learning in a problem-centered environment. *Proceedings of the Twenty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, Ohio:

ERIC. (pp. 1743-1754)

Cazden, C. (1988). *Classroom discourse: The language of teaching and learning*. Portsmouth, NH: Heinemann.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. New York: Teachers College.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, *23*(2):87-103.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Cobb, P., & Bauersfeld, H. (1995). *The emergence of mathematical meaning.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28*(3), 258-277.

Cobb, P., & Yackel, E. (1995). *Constructivist, emergent, and sociocultural perspectives in the context of developmental research.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH.

Confrey, J. (1990). *What constructivism implies for teaching.* Davis, R. B.; Maher, C. A., and Nodding, N., (Eds). Constructivist views on the teaching and learning of mathematics. pp. 107-122. Reston, VA: National Council of Teachers of Mathematics.

Confrey, J. (1995). How compatible are radical constructivism, sociocultural approaches, and social constructivism? In L. P. Steffe & J. Gale (Eds.), *Constructivism in education*. (pp. 185-225). Hillsdale, N.J.: Lawrence Erlbaum.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Donovan, M. S., & Bransford, J. D. (2005). *How students learn: Mathematics in the classroom.* National Research Council. Washington, D.C.: The National Academies Press.

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education, 13*(5), 360-380.

Dreyfus, T., & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness and periodicity. *Focus on learning problems in mathematics, 5*(3-4), 119-132.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky, & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy. MAA Notes,* 25 (pp. 85-106). Mathematical Association of America.

Dugdale, S. (1993). Functions and graphs: Perspectives on student thinking. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Editors), *Integrating research on the graphical representation of functions* (pp. 101-130). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Ernest, P. (1996). Varieties of constructivism: A framework for comparison. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 335-350). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Esty, W. (2005). Teaching about inverse functions. *The AMATYC Review*. 26(2):4-10.

Forman, E. (1989). The role of peer interaction in the social construction of mathematical knowledge. *International Journal of Educational Research, 13*, 55-70.

Forman, E. A. (2003). A sociocultural approach to mathematics reform: Speaking, inscribing, and doing mathematics within communities of practice. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research companion to principles and standards for school mathematics* (pp. 333-352). Reston, VA: National Council of Teachers of Mathematics.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education, 30*(2), 148-170.

Franke, M., Carpenter, T., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*, 653-689.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Herbel-Eisenmann, B., & Breyfogle, M.L. (2005). *Questioning our patterns of questioning. Mathematics Teaching in the Middle School, 10*(9), 484-489.

Herscovics, N. (1996). The construction of conceptual schemes in mathematics. L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (editors), *Theories of Mathematical Learning* (pp. 351-380). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Hershkowitz, R., & Schwarz, B. (1999). The emergent perspective in rich learning environments: Some roles of tools and activities in the construction of sociomathematical norms. *Educational Studies in Mathematics, 39*, 149-166.

Hiebert, J. (1992). Reflection and communication: cognitive considerations in school mathematics reform. *International Journal of Educational Research, 17*(5), 439-456.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., et al. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Karabenick, S. A., & Sharma, R. (1994). Perceived teacher support of student questioning in the college classroom: Its relation to student characteristics and role in the classroom questioning process. *Journal of Educational Psychology*. 86(1): 90-103.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59-80.

Knuth, E. (2000). Understanding connections between equations and graphs. *The Mathematics Teacher, 93*(1), 48-53.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb, & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 229-270). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lampert, M. (1990). Connecting inventions with conventions. In L. P. Steffe, & T. Wood (Eds.), *Transforming children's mathematics education: International perspectives* (pp. 253-265). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education, 36*(2), 101-136.

Markovits, Z., Eylon, B. S., & Bruckheimer, M. (1988). Difficulties students have with the function concept. In A. F. Coxford, & A. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 43-60). Reston, Virginia: The National Council of Teachers of Mathematics.

Maxwell, J. A. (1996). *Qualitative research design: An interactive approach*. London: SAGE Publications.

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education, 32*(3), 236-266.

Mehan, H. (1979). *Learning lessons*. Cambridge, Mass.: Harvard University Press.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass Publishers.

Middleton, J. A., Lesh, R., & Heger, M. (2003). Interest, identity, and social functioning: Central features of modeling activity. In R. Lesh and H. M. Doerr, (Eds.). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 405-431). Mahwah, N.J.: Lawrence Erlbaum Associates.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. London: SAGE Publications.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Perissini, D. D., & Knuth, E. J. (1998). Why are you talking when you could be listening? The role of discourse and reflection in the professional development of a secondary mathematics teacher. *Teaching and Teacher Education, 14(*1), 107-125.

Perry, W.G. Jr. (1970). *Forms of intellectual and ethical development in the college years: A scheme* . New York: Holt, Rinehart and Winston, Inc.

Pesek, D. D., & Kirshner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education, 31*(5), 524-540.

Petitto, A. (1979). The role of formal and non-formal thinking in doing algebra. *Journal of Children's Mathematical Behavior, 2*(2), 69-82.

QSR. (2002). *N6: Software for qualitative data analysis*. Melbourne, Australia: Author.

Rasmussen, C., Yackel, E., & King, K. (2003). Social and sociomathematical norms in the mathematics classroom. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6 - 12* (pp. 143-154). Reston, VA: National Council of Teachers of Mathematics.

Schwandt, T. A. (1994). Constructivist, interpretivist approaches to human inquiry. In N. K. Denzin, & Y. S. Lincoln (Eds.). *Handbook of qualitative research*. (pp. 118-137). London: SAGE Publications.

Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education, 30*(4), 362-389.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In E. Dubinsky, & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy. MAA Notes, 25* (pp. 59-84). Mathematical Association of America.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15*, 3-31.

Skemp, R. R. (1987). *The psychology of learning mathematics: Expanded American Edition*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics, 33*(3), 259-281.

Smith, J. P. (1996), Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, *27*(4), 387-402.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 55-488.

Stewart, J., Redlin, L,, and Watson, S. (2002). *Precalculus: Mathematics for Calculus. 4th ed*. Pacific Grove, CA: Brooks/Cole.

Stigler, J. W., Fernandez, C., & Yoshida, M. (1996). Traditions of school mathematics in Japanese and American elementary classrooms. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 149-178). Mahwah, NJ: Lawrence Erlbaum Associates.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. London: SAGE Publications.

Swokowski, E. W., & Cole, J. A. (2002). *Precalculus: Functions and graphs* (Ninth ed.). Pacific Grove, CA: Brooks/Cole.

Thompson, J. E. (1946). *Trigonometry for the practical man*. Princeton, New Jersey: D. Van Nostrand Company, Inc.

Tobin, K. (2000). Interpretive research in science education. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 487-512). Mahwah, New Jersey: Lawrence Erlbaum Associates.

van Oers, B. (1996). Learning mathematics as a meaningful activity. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 91-114). Mahwah, New Jersey: Lawrence Erlbaum Associates.

van Zee, E., & Minstrell, J. (1997). Using questioning to guide student thinking. *The Journal of the Learning Sciences , 6*(2), 227-269.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education in Science and Technology, 14*(3), 293-305.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics, 34*(2), 97-129.

Voigt, J. (1996). Negotiation of mathematical meaning in classroom processes: Social interaction and learning mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 21-50). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Walen, S. B. (1994). Identification of student-question teacher-question student-response pattern: Students' interpretation of empowerment. *Annual Meeting of the National Countil of Teachers of Mathematics* (ERIC Document Reproduction Service ED372941)

Washington State Board for Community and Technical Colleges. (2005). *Annual year report 2004-05.* Retrieved August 4, 2006, from http://www.sbctc.ctc.edu/data/acadyrrpts.asp

Wertsch, J.V., & Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education.* (pp. 159-174). Hillsdale, N.J.: Lawrence Erlbaum.

Wood, T. (1998). Alternative patterns of communication in mathematics classes; Funneling or focusing? H. Steinbring, M. G. Bartolini Bussi, & A. Sierpinska (Editors), *Language and Communication in the Mathematics Classroom* (pp. 167-178). Reston, VA: National Council of Teachers of Mathematics.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education, 30*(2), 171-191.

Woods, P. (1992). Symbolic interactionism: Theory and method. In M. D. LeCompte, W. L. Millroy, & J. Preissle (Eds.), *The handbook of qualitative research in education* (pp. 337-404). San Diego: Harcourt Brace Jovanovich, Publishers.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Yackel, E., & Rasmussen, C. (2002). Beliefs and norms in the mathematics classroom. G. C. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 313-330). London: Kluwer Academic Publishers.

Yackel, E., Rasmussen, C. & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *Journal of Mathematical Behavior*, 19, 275-287.

APPENDIX A

TEACHER CONSENT FORM

WASHINGTON STATE UNIVERSITY
TEACHER CONSENT FORM
RESEARCH ON INTERACTIONS IN MATHEMATICS CLASSES

Researcher: Jacqueline Coomes, Department of Teaching and Learning, WSU, Pullman

Researcher's statement

I am asking you to be in a research study. The purpose of this consent form is to give you the information you will need to help you decide whether to be in the study or not. Please read the form carefully. You may ask questions about the purpose of the research, what I would may ask you to do, the possible risks and benefits, your rights as a volunteer, and anything else about the research or this form that is not clear. When I have answered all your questions, you can decide if you want to be in the study or not. This process is called 'informed consent.' I will give you a copy of this form for your records.

PURPOSE AND BENEFITS

The purpose of the study is to examine community and interactions in college-level mathematics classes. It is hoped that the results will extend our understanding of the development of communities and interactions and the relationships between classroom dynamics and student learning.

PROCEDURES

Data collected will consist of student answers to a questionnaire, notes and videotape from observations of the classroom, interviews with the teacher, interviews with selected students, and examples of student work, and will preserve the confidentiality of all participants. Interviews may be up to an hour each. You may, at any time, ask questions of me, refuse to answer a question, or end the interview without penalty. The data I collect will be strictly confidential. Any articles from the research will use pseudonyms for the school, teachers, and students. I will tape-record interviews and transcribe the data. Since this research is a project consisting of data from one quarter, all data will be collected before the end of August 2005.

RISKS, STRESS, OR DISCOMFORT

All data will be kept confidential. Raw data will be kept in a locked filing cabinet in a locking private office for five years and then destroyed. There is a possibility of stress or discomfort from being observed and interviewed.

Printed name of researcher Signature of researcher Date

Participant's statement

This study had been explained to me. I volunteer to take part in this research. I have had a chance to ask questions. If I have general questions about the research, I can ask the researcher listed above. If I have questions regarding my rights as a participant, I can call the WSU Institutional Review Board at [509] 335-9661. This project has been reviewed and approved for human participation by the WSU IRB. I will receive a copy of this consent form.

Printed name of participant Signature of participant Date

APPENDIX B

STUDENT CONSENT FORM

WASHINGTON STATE UNIVERSITY
STUDENT CONSENT FORM
RESEARCH ON INTERACTIONS IN MATHEMATICS CLASSES

Researcher: Jacqueline Coomes, Department of Teaching and Learning, WSU, Pullman
<u>Researcher's statement</u>
I am asking you to be in a research study. The purpose of this consent form is to give you the information you will need to help you decide whether to be in the study or not. Please read the form carefully. You may ask questions about the purpose of the research, what I would may ask you to do, the possible risks and benefits, your rights as a volunteer, and anything else about the research or this form that is not clear. When I have answered all your questions, you can decide if you want to be in the study or not. This process is called 'informed consent.' I will give you a copy of this form for your records.

PURPOSE AND BENEFITS

The purpose of the study is to examine community and interactions in college-level mathematics classes. It is hoped that the results will extend our understanding of the development of communities and interactions and the relationships between classroom dynamics and student learning.

PROCEDURES

Data collected will consist of student answers to a questionnaire, notes and videotape from observations of the classroom, interviews with the teacher, interviews with selected students, and examples of student work, and will preserve the confidentiality of all participants. Interviews may be up to an hour each. You may, at any time, ask questions of me, refuse to answer a question, or end the interview without penalty. The data I collect will be strictly confidential. Any articles from the research will use pseudonyms for the school, teachers, and students. I will tape-record interviews and transcribe the data. Since this research is a project consisting of data from one quarter, all data will be collected before the end of August 2005.

RISKS, STRESS, OR DISCOMFORT

All data will be kept confidential. Raw data will be kept in a locked filing cabinet in a locking private office for five years and then destroyed. There is a possibility of stress or discomfort from being observed and interviewed.

Printed name of researcher Signature of researcher Date

<u>Participant's statement</u>
This study had been explained to me. I volunteer to take part in this research. I have had a chance to ask questions. If I have general questions about the research, I can ask the researcher listed above. If I have questions regarding my rights as a participant, I can call the WSU Institutional Review Board at [509] 335-9661. This project has been reviewed and approved for human participation by the WSU IRB. I will receive a copy of this consent form.

Printed name of participant Signature of participant Date

APPENDIX C

RESEARCH TIMELINE

Timeline for research and writing

April 1 – June15: Confirm access; finish writing proposal

June 15 – June 26: D1; IRB approval and community colleges approval

June 20-26: Initial teacher interviews

June 27: [First day of class]: Ask students for permission; have them sign consent forms;

June 27 – August 18: Collect and analyze data, write analytic memos, initial data analysis

August 19, 2005–September 2006: Analyze and write

APPENDIX D

STUDENT QUESTIONNAIRE

Student Questionnaire

1. What is your major?
2. What is your gender? Ethnicity? Age?
3. What is the highest mathematics class you plan to take in college?
4. How do you best learn mathematics?
5. During math class, what are some things a teacher can do that help you learn?
6. During math class, what are some things you do that help you learn?
7. During math class, do you like to work with a partner or in a group? Why or why not?
8. Do you like it when a math teacher involves the class in discussion? Why or why not?
9. Do you usually offer input during class discussions? Why or why not? What kinds of input do you usually offer [ask questions, make suggestions…]?
10. Does listening to other students' questions or explanations help you learn? Explain.
11. Do you find that memorizing steps and formulas is important in mathematics? Explain.
12. When learning new math concepts, what do you do to understand the ideas, where the formulas come from, and why you take the steps you do?
13. Do you like to get to know your teacher and/or other students in the class?

APPENDIX E

TEACHER INTERVIEW PROTOCOL

Teacher Interview Protocol

1. What is your philosophy of teaching and learning and how did you develop this philosophy? [beliefs about student learning]
2. Is your philosophy different for different classes [precalculus vs. math reasoning or finite or calc III?]
3. What constraints are imposed by the department, school, or other concerning the text you use, the content you cover, and the testing and grading procedures?
4. What do you like about teaching? [Listen for what they value in the environment and interactions.]
5. In thinking back to one of your best classes [over a quarter], what made it successful? In thinking about one of the worst classes, what made it unsuccessful?
6. What are the characteristics of students who in perform well in you class?
7. What can you say about the characteristics of students who are apt to fail your class?
8. Describe the kind of environment you like to establish in your classroom. Why? What role do students play in developing this environment? Do all students participate right away? [Listen for characteristics of environment: social norms, students' identities, values, beliefs, authority, activities].
9. Is the nature of the environment you establish different when the class is much larger? How?
10. How would you describe the types of interactions you strive for with your students? Why do you foster these interactions? Can you describe times when students interacted the way that you intended? Can you describe times when students did not interact the way that you intended? How did you handle those situations?

11. How would you describe the type of relationship you prefer to have with your students? What do students gain? Do some students resist?
12. What do you do, if anything, to help students get to know each other? What, if anything, do you think students gain by knowing each other?
13. Do students in different courses [say elementary algebra, math reasoning, or calculus, as opposed to precalculus] respond in different ways to your efforts to establish environments, interactions, and relationships? If so, how?
14. What characteristics do you think a student needs to have to enter calculus and be successful? [These are the characteristics we want students to leave precalculus with.]
15. Do you have any particular goals for these students based on what math class they will be taking next?

APPENDIX F

STUDENT INTERVIEW PROTOCOL

Student Interview Protocol

This protocol will be used throughout the quarter with individual students to gather data concerning students' perceptions about the interactions and to understand the extent to which the interactions are influencing their ways of knowing. In addition, I will photocopy their classroom notes.

1. What do you think you are expected to know from this day's lesson?
2. When you go back and look at your notes, what will you look for?
3. What connections was the teacher trying to make? Did you understand these connections? How will these connections help you understand the new concepts?
4. When the teacher [or other student] explained … did you think it was important to try to understand what they were saying? Did the explanation help you understand what to do or why you want to do it? Why or why not? Was it hard or easy to understand? Do you have your own way of understanding? Can you explain to me how you understand it now?
5. What parts of today's discussion will help you learn this material?
6. Were there any parts of the discussion that you did not understand? How do you handle parts you do not understand?
7. When Mr. ___ gave you a problem to work [or task to do; I will describe the specific instance and perhaps find it in their notes for reference during the discussion.], how did you approach it? Were you able to complete it before the class discussion? What insights about the problem did you gain, if any, from the class discussion? Would you do a similar problem the same way again, or would you change your tactic? Why?

APPENDIX G

CLASSROOM OBSERVATION PROTOCOL

Classroom Observation Form

Teacher: Date:

I. Classroom Environment
   a. Physical Setting
   b. Demographics
      i. Number of students
      ii. Ages of students
      iii. Race/ethnicity composition
      iv. Gender composition
      v. Refer to attached seating chart to indicate who is interacting
   c. General description of community:
      i. Mathematical authority [Evidence of who/what has mathematical authority?]
      ii. What roles do students and teacher assume in the community?
      iii. What relationships form in the community?
      iv. What norms and rules structure the activities and interactions?
      v. What beliefs appear to be held about what mathematics is and how it is learned?
      vi. In what ways do the participants either implicitly or explicitly contribute to the development of social norms?

II. Mathematics
   a. Goal of the lesson:
   b. Description of activity or task:
      i. What is the content focus?
      ii. What is the level of cognitive demand as set up?
      iii. What level of cognitive demand is implemented?
      iv. What factors contribute to the support or decline of cognitive demand?
      v. What mathematical practices are used?
      vi. What do their mathematical practices indicate about how they are striving to know the mathematics?

III. Sociomathematical norms:

a. What counts as an *acceptable explanation*?
b. Are *different ways of explaining* solicited or offered, and if so, how are they *used* by the teacher and students?
c. In what ways were *mathematical objects* such as graphs, equations, definitions, proofs, hypotheses, justifications, etc, discussed and used to *reflect* on?
d. Do students offer *elegant or sophisticated* solutions? What is the response from others?
e. How are *errors* handled?
f. If used, what is the nature of students' work *together*?
g. How and when do students participate in *argumentation*?
h. What *learning strategies* do students appear to be employing?
i. Do students initiate sociomathematical norms?
j. During an activity [group or otherwise] what did the teacher do? How did students engage in the activity? Was it what the teacher intended?

IV. Communication
   a. Types
      i. Uni-directional
      ii. Contributive
      iii. Reflective
      iv. Instructive
      v. Telling [Initiating and/or eliciting and/or other]
         1. Teacher's apparent intent
         2. Students' apparent interpretation
         3. Conceptual or procedural
      vi. Transactive [clarify meaning]
      vii. Metacognitive
   b. Who is not interacting? What are they doing?

V. Observer comments:

APPENDIX H

OBSERVATION, MR. REILLY: JULY 13

Observation, Mr. Reilly: July 13

Context: They had solved $\sin 2\beta = \frac{1}{\sqrt{2}}$ for all solutions, then he writes on the board as he talks:

find all solutions in the interval [0, 2pi)

Mr. Reilly: Swokowski [text author] is an evil man; he knows when you get it and when you don’t get it, so he’ll do things like this: find all solutions in the interval; and I’ll use the same question. Because he knows everybody learns by mimicry first, they don’t think about it til later. I’ll start with simply this: how many are there?

S: Three [very softly].

S: Two [very softly].

Mr. Reilly: [Does not appear to hear them] Cuz, see, to answer the question here [walks over to point at the previous example where they had solved the equation for all solutions], the answer is, how many solutions? Zillions, see [he is pointing at the solutions], zillions, why? Cuz they could be anybody. Now, we're in this interval [walks back over to the current example] so you're not going to go round and round and round, you're only going to go around the circle once. How many answers are there when you go around the circle once?

Ss: Two.

Mr. Reilly: Yeah, you're wrong. Swokowski knew that, that's why he asked. He's like I knew you didn't get it, but better to tell you now than to tell you on the test.

Susan: Once.

Natalie: Four.

Mr. Reilly: Yes, why? [she uses her hand to indicate forward one revolution and backwards one revolution.]

Mr. Reilly: No, because backwards is negative. No, no, remember the most common mistake? You forget to divide everybody by two. Here the answers are; everything else is something you were doing to get the answer. You have an answer of pi over eight. I'm not going in jumps of two pi anymore, I'm going in jumps of pi. Everything got contracted, and since I got contracted, I get somebody from third quadrant too. I get pi over eight, I also get nine pi over eight. That stupid two, remember, in the domain it has the opposite affect. Uh, three pi over eight, three pi over eight is right about here. And there's another here, do you know who that is? [He plotted all these solutions on a unit circle as he talked through.]

S: Eleven pi over eight.

Mr. Reilly: Eleven pi over eight. Is that it? Yeah?

Tim: Aren't those, the nine pi over eight, be negative? Make the sine be negative?

Mr. Reilly: No, I'm going forward. Now you're right, I could have gone negative too. What I did is, I went from pi over eight to nine pi over eight by going pi forward.

Tim: But the sign of it, it wouldn't be same sign; it would make it a negative one over the square root of two.

Mr. Reilly: Oh, I see what you're thinking. [This was genuine surprise and appreciation.]

Susan: That's why I thought there was only two too.

Mr. Reilly: Ah, yeah, yeah, tell you what, let's check. I agree, you should be worried about that. And it's a good point: wait a minute, isn't the sine of this negative, and come to think of it, don't all four of these angles have a different sine? The sine of this is that high, the sine of this is that high, the sine of this is that high, the sine of this is that high. Well, this is all screwed up. Well, I'll check one of them. I'll check the one that's negative. Check beta equals nine pi over eight. [Writes $\sin\left(2\cdot\frac{9\pi}{8}\right)$.] The sine of two times nine pi over eight is the sine of nine pi over four. [pause] What quadrant is that? [pause] See, stupid two, messed everything up. You were never really in the third quadrant for sine, because when you double it, you're nine pi over four. What quadrant is that?

Susan, Mark: Three.

Mr. Reilly: First.

Susan: First.

Mr. Reilly: That's the first quadrant. Nine pi over four is here. And the sine of that is,

S: Positive.

S: One over root two.

Mr. Reilly: One over root two. Yeah, you got it in the reverse order, you double the angle first and then you look at the sine value, see double first. So if you double pi over eight, you're here. Now look at the sine of it. Double nine-pi over eight, you're here, now look at the sine of it. If you double three pi over eight, you're here, then you look at the sine. If you double this angle, guess where you are? You're here, you double it, and after you double it, everybody has the same sine value. … I think that's part of what happens to students on the test, if you just see it as a mechanical process, it's really easy to think you've got it down cold, because you go through the motions and you match the back of the book. What you really want to do before you take the test is get really confused. Get the answer right, but confuse yourself with it. Why would four different points on a circle that are at different heights all be a solution to this? Then fight through it. That way you get confused when there's no risk involved.

APPENDIX I

OBSERVATION, MR. REILLY: JULY 21

Observation, Mr. Reilly: July 21

“A cathedral is located on a hill, as shown in the figure. When the top of the spire is viewed from the base of the hill, the angle of elevation is 48º. When it is viewed at a distance of 200 feet from the base of the hill, the angle of elevation is 41º. The hill rises at an angle of 32º. Approximate the height of the cathedral.” (Swokowski & Cole, 2002)

The drawing in Figure 7 is similar to the one Mr. Reilly drew on the board. The points A-E were not labeled in the textbook, and the point E, segment CE and segment DE were not included in his original drawing. [episode starts at 45:55 when he reads the problem and draws and labels the picture on the board. The transcribed part starts at 48:15]

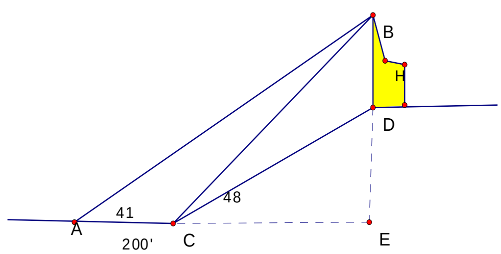

Figure 7. Cathedral problem.

Mr. Reilly: I'm tired, I'm old, I'm tenured, you tell me what to do, I'll follow whatever it is, even if it's wrong.

Susan: Name your points.

Mr. Reilly: Name your points, I like that. Uh, cool. A, B, C, D, good enough? [pause 4 seconds] Okay, silence means yes by the way.

Susan: Okay, yes.

Mr. Reilly: And then?

Tim: Subtract one eighty from forty-eight.

Mr. Reilly: Okay, you want to get this angle here, and that little guy is one thirty two? [he draws it, apparently inferring from the suggestion where it should be.]
Susan: Yeah.
Mr. Reilly: All right?
Natalie: Now you can get the top one at B.
Mr. Reilly: Okay, I just have to ask: So?
Natalie: So you can go get the sine.
Mr. Reilly: I mean do you have a plan?
Natalie: Yeah, to get the sine.
Mr. Reilly: Well, I just want to know because I am lazy; I don't want to just go find an angle because I can.
Susan: Then you can use the law of sines.
Mr. Reilly: Oh, okay, so you want to go get this little guy here? And once you've got that little guy there, you've got?
Natalie: The side.
Mr. Reilly: You've got your ASA, which means you can find anybody on ABC, who was it you were planning on finding?
S: BC.
Mr. Reilly: BC? Okay, here let me write this down, because like I said, I've got a bad short-term memory. [starts writing a list to the left of the drawing as he talks] So you want to go, you've got your one thirty two, so you want to find angle ABC and then use the law of sines to go get BC. Okay, once you get BC? [pause 4 seconds]
Natalie: Find AC.

Reggie: Find the little tiny, the little angle, forty-eight from thirty-two, you can get that.

Mr. Reilly: BCD?

Reggie: Yeah.

Mr. Reilly: [writing on the board on the list he is making] Okay, get BC, get angle BCD.

Susan: Sixteen, sixteen degrees.

Mr. Reilly: Sixteen degrees, okay [he writes it on the drawing] [pause 7 seconds] You see this is what I was worried about.

Susan: Well, take your forty-eight-degree angle and subtract it from ninety to get,

Natalie: Get the side first.

Mr. Reilly: [points to Reggie] What you got?

Reggie: I had another triangle.

Mr. Reilly: What'd you draw?

Reggie: From point D straight down at a ninety degree [is motioning with his hand].

Mr. Reilly: [as he draws it and labels the point] Okay, I think I'll call that E.

Reggie: Okay, we know the angle at D now.

Mr. Reilly: Okay, good, get CDE, I'll write this down here, get angle CDE [writes it on his list, then goes to drawing]. Okay, so I've got that one, I've got, okay, I've got that one. See I'm marking the ones that I can go get.

S: CD.

Mr. Reilly: CD, really?

Susan: Well you can do the law of sines to get line AB, or, yeah, the distance from the top to the bottom of the hill.

Reggie: You can get the angle.

Mr. Reilly: The angle?

Reggie: Yeah, the angle at point D, the obtuse angle right there.

Mr. Reilly: Oh, okay.

Reggie: And then from there you can get the angle at the very top of the cathedral and then you can figure out *h*.

Mr. Reilly: Okay, now I'm starting to feel better about this. All right, what I think I hear you saying is, the ultimate strategy is an ASA [marks it on the drawing]. That's the ultimate strategy, is an ASA. To get my little hands on that ASA, I've got sixteen, I figured that out. Uh, by the way, there's lots of ways to do this problem; there are lots of ways, with a right triangle, without a right triangle. There are lots of ways to do this. Um, I'm going to take the first one that works, though, okay. Uh, I got my sixteen, so I got my first A. Somebody, okay we already figured out that I could get BC, following this [indicates his list], and then how hard is it going to be to get this angle up here? [indicating angle CBD]

[He goes on to talk about how much he likes this problem because their "life is a lot better because they have to draw a picture." Then he recaps how the drawing of the vertical line suggested by Reggie helped by creating several right triangles.]

Mr. Reilly: And then you come back and say, how does it sound? Does it sound right? How tall is that? Is that about right for a spire on a church? On a cathedral?

Ss: It's pretty tall. [ends at 57: 35]

APPENDIX J

OBSERVATION, MR. ANDERSON: JULY 18

Observation, Mr. Anderson: July 18

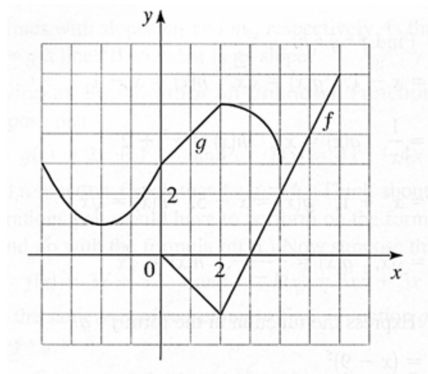

Figure 8. Graphs of Two Functions.

“Use the given graphs of *f* and *g* to evaluate the expression:

23. $(f \circ g)(0)$ 24. $g(f(0))$ 25. $(g \circ f)(4)$ 26. $(f \circ g)(0)$” (Stewart et al., 2002, p. 225). (From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)

Daniel: I had a question: how do you even, what they even want out of twenty-three through twenty-eight? (see Figure 8)

Mr. Anderson: Okay, twenty-three through twenty-eight. Oh, they want a numerical value.

Daniel: Okay.

Mr. Anderson: Like, looking at the graph there, they have a graph picture of *f* and a graph picture of *g*.

Daniel: Right.

Mr. Anderson: They don't have a, like this is some function [pointing to an algebraic rule still on the board] written out in a formula. They don't have a formula for it. So, first off, let's just look at *f* for a minute; what is *f* of one? [more to the whole class] We're looking at page two hundred and twenty-five, that red line there is the graph of *f*. The question is, what is *f* of one?

Sarah [and another student]: Negative one.

Mr. Anderson: Negative one. What is *f* of two?

Sarah: Negative two.

Mr. Anderson: Negative two. Why is *f* of one negative one and *f* of two negative two? [pause] Those are correct. What does *f* of one mean?

Sarah: When *x* is one.

Mr. Anderson: [writes $f(1)$ on the board and points] *f* of one means *x* is one, find the *y*-value. [pause 3 seconds] What's the *y*-value on the *f* function when *x* is one? It is negative one. We're just trying to get used to graphical pictures of functions, here. What is *f* of two in that picture?

S: Negative two.

Mr. Anderson: Negative two. What is *f* of three?

S: Zero.

Mr. Anderson: No, not zero [looks at his book] Oh yeah, you're right. It came up [inaudible] Sorry about that; I thought it came up at the same slope, but it's steeper. How about *f* of four?

Daniel [and others]: Two.

Mr. Anderson: Yeah, so you're getting used to what they're telling you with these functions.

Daniel: Okay.

Mr. Anderson: So the points on function graphs actually tell you an input and an output value. So, we usually graph by, like if I give you a function, and if I give you *f* of *x* equal *x* squared, [inaudible] by graphing, you'd make your table and then you'd go graph. We're kind of going backwards here.

Daniel: So you find *g*, if it's two, you go plug it into the *f*?

Mr. Anderson: Exactly, so, uh,

Daniel: I can do it now, so,

Mr. Anderson: Yeah, let's do one for practice.

Carol: Yeah, do it; I'm not sure [inaudible].

Mr. Anderson: Let's try twenty-four. Okay, twenty-four says g of f of zero. Oh, that's not a very interesting one, oh well. So start on the inside; what is f of zero?

Ss: Zero.

Mr. Anderson: Zero. So we need g of zero, which is,

Ss: Three.

Mr. Anderson: Three, so they're just getting you used to function notation, but they're giving you information in a different way than you're used to seeing it. You're used to seeing it this way. They're giving it to you graphical picture. But the graphical picture contains the same information, but a different way of representing it. Let's see, what else?

Sarah: What about twenty-three?

Mr. Anderson: Uh, twenty-three, f of g of two, sure. So, twenty-three, f of g of three?

Sarah: Two.

Mr. Anderson: Two. What do we start with, the f function or the g function?

Ss [including Sarah]: The g.

Mr. Anderson: g, right. Start on the inside; you need to find g of two; g of two looks like it's [pause]

S: Five.

Mr. Anderson: Five. So g of two,

Sarah: Oh.

Mr. Anderson: Is five.

S: Okay.

Mr. Anderson: Yeah, the *x*-value is two, we're using the g function, the *y*-value is five. Now look at your picture and find *f* of five. [looking at his book] Two, three, five; it looks like four. [Carol and Sarah are speaking quietly to each other.]

Carol: Can you do twenty-five too? [inaudible] twenty-six.

Mr. Anderson: Sure, yeah, uh, [erases part of the board] twenty-five; *g* composed with *f*. Well, what does *g* composed with *f* mean? [pause 2 seconds; points to the expression he wrote as he said it] What does this mean?

Daniel: It's the same as that.

Mr. Anderson: Is it the same as this one or this one? [pointing to two of the earlier expressions]

Daniel: The one on the right. [Carol is also saying something inaudible]

Mr. Anderson: Yeah, *g* of *f*. So the first thing to do when you see it written this way is just rewrite it in the way that makes more sense [writes $g(f(4))$].

Carol: Yeah, now it looks better; now I've got it.

Mr. Anderson: You can get it from there? So, what would this one be, *f* composed with *g* of three?

Carol: *f* of *g* of three?

Mr. Anderson: *f* of *g* of three. I don't know if that works out, but,

S: Well, so twenty-six would be *f* of *g* of zero?

Mr. Anderson: Exactly. Now sometimes you can do the first piece but not the second piece. Um, [looking at the page in his text] let me give you an example. How about *g* composed with *f* of, uh, six. So, I'm doing *g* composed with *f* of six.

Daniel: Doesn't work.

Mr. Anderson: Right, because if you try, it'd be *g* of *f* of six, *f* of six is no problem, [looking at the book] *f* of two, four, six, is six. This *f* of six is six.

Daniel: There's nothing to relate it to on,

Mr. Anderson: Right, but there is no point on the *g* graph with *x*-value six. What do we write?

Ss: No solution.

Mr. Anderson: Not no solution, we aren't solving an equation.

Carol: I put no point on the graph.

S: Undefined.

Mr. Anderson: Undefined, yeah, [points to Carol] What did you put? No point on the graph? That works. [She laughs] Or, six is not in the domain of this composition function.

APPENDIX K

OBSERVATION, MR. ANDERSON: JULY 27

Observation, Mr. Anderson: July 27

Daniel: Fifty-three.

Mr. Anderson: Fifty-three, okay. So, we have, our polynomial is [says it as he writes it; the polynomial is $P(x) = x^5 - 3x^4 + 12x^3 - 28x^2 + 27x - 9$] All right, our job is to find all zeros. So, if I was doing this just to find all zeros, I would start out with a graph. So, let's get a graph [28 seconds pass as he puts it in his calculator quietly; Most students also put it in their graphs except Sam.] And then you gotta use your graph to try to figure out what's going on. Where does it cross the *x*-axis? [pause 8 seconds] Most people are still typing it in [pause 15 seconds]. Got a guess?

Daniel: Well, it looks like it follows the *x* for the,

Mr. Anderson: Okay, so let me give you a different window so that you can see it a little bit better.

Daniel: Because I'm at negative two, two on the *x* and it looks like its,

Mr. Anderson: Yeah, it follows along [Sam is getting his calculator out now.]

Daniel: From point seven five to one point two five.

Mr. Anderson: But it still crosses over at one particular point, and you can see it a little better if you do the graph from, uh, zero to two and from negative point two to point two on the *y*'s. It's still kind of flat there, but it only crosses at one point.

Daniel: So it looks like one.

Mr. Anderson: It looks like one. So, it looks like one, so what we're going to do is one, then [sets up synthetic division] divide it out and see what we have left. Okay, so this part, just going to crank through it [does arithmetic silently] and, sure enough, it works. Okay, so we have one as a zero. But now looking at your graph there, it looks like, well, there are no other graphs, no other *x*-intercepts. Well, let's think about it for a minute, here. Because what we now know, you know, when we're factoring, we'd have an *x* minus one here, maybe some other stuff. Okay, now we're used to just getting a power of one there. But maybe it's not one, what are the other possibilities?

Daniel: Squared, cubed.

Anthony: Cubed.

Mr. Anderson: Squared, cubed, and Anthony says cubed but not squared

Anthony: It's not squared.

Mr. Anderson: Why not squared?

Anthony: It would touch the *x*-axis.

Mr. Anderson: It would touch it and come down if it were squared.

Daniel: Oh, parabola.

Mr. Anderson: So, we actually have three possibilities [he writes a 3 and 5 above the 1 exponent on the factor $(x-1)$].

Daniel: Cubed or fifth-root.

Mr. Anderson: Yeah, we can't get higher than that because we're in the fifth degree. So if you don't have any other zeros.

Daniel: It must be the fifth-root.

Mr. Anderson: Then it must be one of these two, not just this one. So, in other words, let's try dividing out one again. We're dividing out *x* minus one again. And then, the trick is, you know how you commented that it was following along the *x*-axis? The more it does that, the flatter it is there, usually, the higher the degree on that zero, if you're on the factor portion of the zero. So that flat part is actually a key; it kind of indicates that it's probably one of these other ones than a one. That's something we hadn't talked about, so I understand [quietly works through the synthetic division]. And there it is; it worked twice. But it can't be two, so it must work one more time [quietly works through synthetic division again]. And you could work through it one more time after that, but I don't think it's going to work. So, we know it's not one, it's either three or five now up here. But if you tried it again, just real quick, let's try one more time [quietly works through synthetic division again]. Nope, didn't work, Okay, so we don't really want that [indicates last one]. I did it in a different color, so ignore that. So what's the exponent have to be here?

Ss: Three.

Mr. Anderson: Three. Now, what's left as a multiplier? [several students respond inaudibly] This thing, what's this say [pointing to last line on synthetic division]?

Ss: *x*-squared plus nine.

Mr. Anderson: *x*-squared plus nine. Now that's not going to give real zeros which is why there's no other place that it crosses that we can see. So find the other zeros, we just need to look at *x*-squared plus nine. Looking at this portion here [points to the factor $x^2 + 9$]. So *x*-squared equals negative nine.

Daniel: Three *i*.

Mr. Anderson: Yes, plus or minus three *i*. So summarizing, our zeros are one, three times. So you can either write it three times or you can say with multiplicity three, or you could just put one; I'm not really too worried about that. I'm going to write it three times, and then three *i*, negative three *i*. Those are our zeros. [Points to factored form of polynomial] This is not completely factored yet. This right here, if you multiply these you would get this. To be completely factored, we would have to have x minus one, cubed. Does anybody know what else we need?

Sarah: x minus,

Mr. Anderson: [Pointing to her] x minus,

Sarah: Three, x plus three.

Mr. Anderson: Close, three,

Anthony: i.

Mr. Anderson: i, and x plus three i. The completely factored form looks like this. Where's that coming from? [Pointing to list of zeros] That's x minus this, x minus this, x minus this, x minus this, x minus this, and x minus this, all multiplied together. That's what this says. Yeah, that was a good problem. Questions on that one? So, if it crosses over but it's flat, then chances are it's a higher power on the factor than one. Which means if it crosses over it has to be at least three. [10-second pause] Other questions this section? [They spent nine minutes on this problem. There is silence for another 47 seconds before Carol speaks up.]

APPENDIX L

OBSERVATION, MR. ANDERSON: AUGUST 3

Observation, Mr. Anderson: August 3

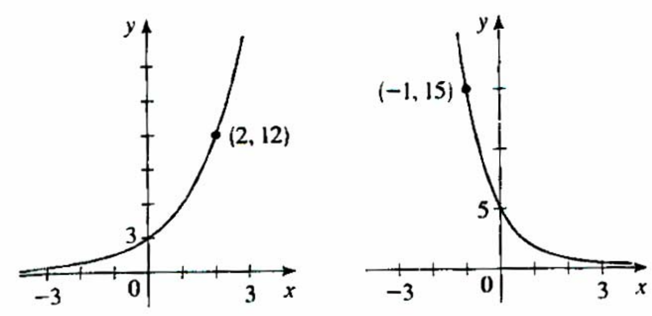

Figure 9. Exponential graphs.

(From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)

Mr. Anderson: How about thirty-three; did you get that one? They give you a graph and want you to find the equation. No, you didn't get that one? Let's take a look at thirty-three. Now they give you two things, there. They give you this point, zero, three, and they give you this point, I think it's two, twelve. And they say that this function is something like *c*, *a* to the *x*. So there's two things you gotta find, you gotta find the *c*; you gotta find the *a*. All right, well if it were just this, for a minute, *a* to the *x*, the graph would be the same as this one here [indicating graph of #25 which is still on the board] you know, our regular stuff, because the *y*-intercept is zero, one. Ignoring that graph for a second, if we have this graph here, what does this *c* do? [He has drawn the graph and labeled the points.]

S: Vertical stretch.

Mr. Anderson: Stretch, yeah, vertical stretch. So we stretched it and we went from zero, one, and we ended up at zero, three.

Daniel: Plus two.

Carol: Times, times.

Mr. Anderson: Oh, that would be a shift. If it was a shift that would have been good, but,

Carol: Multiplication; three.

Mr. Anderson: Multiplication, so it's a three. So the *c* value is three. So, for our equation, we just figured out that the *c*-value is three [pause 8 seconds]. So now, we know our function is three *a* to the *x*.

Kenny: *a* is two.

Mr. Anderson: And *a* is two? How'd you get that?

Kenny: Uh, I used twelve.

Mr. Anderson: Okay, [points to the labeled point on the graph] well twelve, if we get twelve right here, we're supposed to get a two right there [writes $12 = 3a^2$]. So what would you square, then multiply by three to get twelve? Or, differently, divide by three, *a* squared is equal to four, so *a* equals two. So, our equation is *y* equals three times two to the *x*. All right, so try thirty-four. [He stands at the front looking at his book while they work quietly for 90 seconds. Then he starts walking around and I hear his side of each conversation. I can hear that a student has asked him a question.] Well, five *a* to the, no, *a* to the? Nope, not two, [pause 4 seconds] Why did I use two here? I didn't use two just because [inaudible]. Where did this two come from? Where's the twelve? There you go. [He has another similar conversation with another student.]

Mr. Anderson: [to Daniel] Wait, now why did you put squared on that one?

Daniel: You said *a* squared is four.

Mr. Anderson: Well, where did the square come from on that one? Why did I use a two there on that one?

Daniel: Oh, because of the, okay, so that should be negative one.

Mr. Anderson: Exactly, yeah, we’re not going to use a two every time. [To another student] Why is this one squared here? I have no idea why we have a two there. I have no idea why it would be. Is there something there that says we need a two for *x*? Sure, but my question is where did that two come from? Well, in general it’s *x*, … There you go. [More students are talking to each other so it is hard to hear conversations. Carol explains something to Daniel] So, you gotta use this point here… What does *a* to the negative one mean? … [to the whole class] So, five times,

Carol: One-third.

Mr. Anderson: One-third to the,

Carol: *x*.

Mr. Anderson: *x*.

Kenny: I got five times three to the negative *x*.

Mr. Anderson: That is actually the same equation!

APPENDIX M

IN-CLASS PROBLEMS FROM TEXT, MR. ANDERSON

Section 1.6 [June 28] (From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. p. 71-74. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)
Directions: Use the principles described in this section to answer the question posed.

12. The sum of the squares of two consecutive even integers is 1252. Find the integers.

24. During his major league career, Hank Aaron hit 31 more home runs than Babe Ruth hit during his career. Together they hit 1459 home runs. How many home runs did Babe Ruth hit?

36. A box with a square base and no top is to be made from a square piece of cardboard by cutting 4-in. squares from each corner and folding up the sides, as shown in the figure [see Figure 10]. The box is to hold 100 in^3. How big a piece of cardboard is needed?

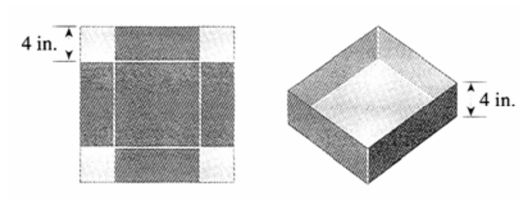

Figure 10. Box problem.

(From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)

48. A woodcutter determines the height of a tall tree by first measuring a smaller one 125 ft away, then moving so that his eyes are in the line of sight along the tops of the trees, and measuring how far he is standing from the small tree [see Figure 11]. Suppose the small tree is 20 ft tall, the man is 25 ft from the small tree, and his eye level is 5 ft above the ground. How tall is the taller tree?

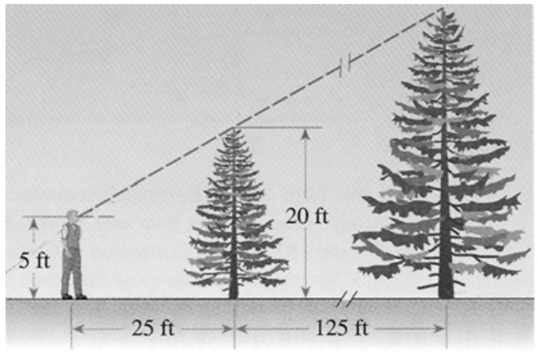

Figure 11. Tree problem.

(From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)

60. After robbing a bank in Dodge City, the robber gallops off at 14 mi/h. Ten minutes later the marshal leaves in hot pursuit at 16 mi/h. How long does it take the marshal to catch up with the bank robber?

Section 1.10 [July 5] Stewart et al., 2002. From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. p. 124-127. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)

14. Find an equation for the line whose graph is sketched.

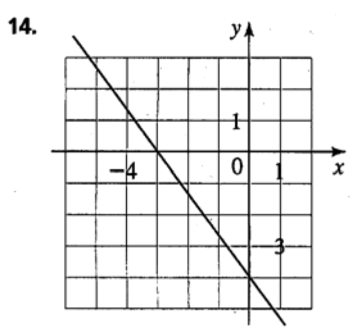

Figure 12. Graph of a line.

(From Precalculus: Mathematics for Calculus, 4th edition by STEWART/REDLIN/WATSON. 2002. Reprinted with permission of Brooks/Cole, a division of Thomson Learning: www.thomsonrights.com. Fax 800 730-2215.)

28. Find an equation of the line that satisfies the given conditions: y-intercept 6; parallel to the line $2x + 3y + 4 = 0$

42. Find the slope and *y*-intercept of the line and draw its graph. $2x - 5y = 0$

56. Find the area of the triangle formed by the coordinate axes and the line $2y + 3x - 6 = 0$.

70. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.

(a) Assuming that the relationship between cost and the number of chairs produced is linear, find an equation that expresses this relationship. Then graph the equation.

(b) What is the slope of the line in part (a), and what does it represent?

(c) What is the y-intercept of this line, and what does it represent?

Table 1

Ways of Knowing (Baxter Magolda, 1992; Brew, 2001)

Ways of knowing	Description	Roles of learner, instructor, peers
Silence (Belenky, et al., as cited in Brew, 2001)	It is neither possible nor important to understand.	Learner: Follow rules, stay quiet. Peers: Cannot learn from peers, so no role. Instructor: Teacher is authority; must show exactly what to do.
Absolute	Knowledge is certain and provided by authorities. Two types (descriptions follow).	
Absolute: Receiver	Knowledge is certain and provided by authorities; minimal interaction with instructor; prefers comfortable environment; relationships with peers; and opportunities to demonstrate knowledge.	Learner: listen and record information. Peers: Talk to create a relaxed environment. Instructor: Facilitate students' reception of information.

Table 1 (continued).

Ways of knowing	Description	Roles of learner, instructor, peers
Absolute: Mastery	Knowledge is certain and provided by authorities; prefer a verbal approach; critical of instructors; expect interactions that aid mastery of knowledge.	Learner: Participate in interesting activities, show instructor student is interested. Peers: Quiz or debate to further learning. Instructor: Use interesting methods.
Transitional: Interpersonal	Believe some knowledge is uncertain (often contextual: e.g. humanities uncertain, chemistry certain). Interpersonal: focuses on uncertainty; resolves by personal judgment.	Learner: Collects others' ideas. Peers: Offer new ideas. Instructor: Creates rapport with students and allows student involvement and self-expression.
Transitional: Impersonal	Balances certainty and uncertainty of knowledge; resolves by logic and research.	Learner: Strives to understand rather than memorize; exchanges views. Peers: Express opinions. Instructor: Focus on understanding and challenge students to think.
Independent knowing: Two types: Interindividual and individual. Similar to Procedural (Belenky et al., 1997)	Knowledge is uncertain. However, this perception of knowledge may be contextual since examples were given of students who disliked classes for which they believed knowledge was certain such as physics.	Learner: Thinks for themselves; shares views with others; creates their own perspective. Peers: Share views; serve as a source of knowledge. Instructor: Promote independent thinking and exchange of opinions.
Contextual knowing	Knowledge is uncertain but judgments are possible based on context.	Learner: Integrates and applies knowledge; thinks through problems. Peers: Contribute quality ideas. Instructor: Promote application of knowledge in context, evaluative discussion of perspectives.

Table 2

Sociomathematical Norms

Sociomathematical norm	Description	Opportunities for student learning
Characteristics of an acceptable explanation (Kazemi and Stipek, 2001; Rasmussen et al., 2003; Yackel and Cobb, 1996).	Explanations must be based on actions taken on experientially real objects; Other students must be able to interpret (Rasmussen et al., 2003; Yackel and Cobb, 1996).	Students must consider how others will interpret their explanation, so they must reflect on their explanation; learn to communicate mathematically (Cobb et al. 1997).
Nature of mathematical thinking involves understanding relations among multiple strategies (Kazemi and Stipek, 2001).	Mathematical thinking involves understanding relations among multiple strategies (Kazemi and Stipek, 2001).	Opportunities to make connections.
Characteristics of a different solution (McClain and Cobb, 2001).	Solutions must be mathematically different (Kazemi and Stipek, 2001; Yackel and Cobb, 1996).	Solutions become objects of reflection as students determine if they are *mathematically* different.
Reification of mathematical objects (Cobb et al., 1997).	Shifts in which what the students and teacher do in action subsequently become explicit objects of discussion.	Supports mathematical practices of exploring, mathematizing, framing questions, use of notation.
Value and identification of easy, sophisticated, or elegant solutions (McClain and Cobb, 2001).	Tacit or implicit establishment; Representative of mathematical community (McClain and Cobb, 2001).	Students try new methods of solving problems but only when they understand them (Lampert, 1990).
Use of errors (Borasi, 1994; Kazemi and Stipek, 2001).	Entire class investigates contradictions (Kazemi and Stipek, 2001).	Conceptual refinement, accommodation, learn from disequilibrium.

Table 2 (continued)

Sociomathematical norm	Description	Opportunities for student learning
Expectations of small-group collaboration (Goos et al., 2002; Kazemi and Stipek; 2001; Yackel and Cobb, 1996).	Students use transactive reasoning within the group; Accountability: All students must be able to give the explanation for the group, to give reasons for agreeing or disagreeing.	Students must listen to and strive to understand the solutions of others (Rasmussen et al., 2003; Yackel and Cobb, 1996).
Expectations regarding argumentation (Wood, 1999).	Knowing how and when to participate; students must listen to and make sense of each others' arguments (Wood, 1999).	When students experience conflict with their prior ways of knowing, they have an opportunity to learn.
The role of computer representations (Hershkowitz and Schwartz, 1999).	The tool is a way to refute or confirm ideas.	Students have a chance to rethink their ideas when the graphics calculator refutes their conjecture.

Table 3

Types of Communication

Type of communication	Description
Uni-directional	Teacher dominated by lectures; questioning limited to closed questions (Brendefur and Frykholm, 2000).
Contributive	Interactions are limited to assistance or sharing and are typically corrective in nature (Brendefur and Frykholm, 2000).
Reflective	Students share ideas, strategies, and solutions, and use these as springboards for deeper investigations and explorations (Brendefur and Frykholm, 2000).
Instructive	Teachers pose situations that lead to modification of students' mathematical understanding and help teachers understand students' thought processes (Brendefur and Frykholm, 2000).

Table 4

Student Demographics: Mr. Reilly's Class

Name	Major	Gender	Ethnicity	Age
Karen	Pre-pharmacy	F	W	27
Susan	(no response)	F	(no response)	20
Nick	Engineering	M	W	19
Reggie	Engineering	M	W	19
Owen	MET	M	W	26
Jake	Aerospace engineering	M	W	19
Steve	Computer engineering technology	M	W	32
Jeremy	Construction management	M	W	25
Tim	Construction engineering	M	W	23
Ryan	Finance/ Econ	M	Native American	22
Mark	Engineering	M	Grebo	27
Julie	Engineering	F	W	35
Shawna	Sports med.	F	W	28
Natalie	Education- math	F	W	42

Table 5

Demographics: Mr. Anderson's Class

Name	Major	Gender	Ethnicity	Age
Sarah	Secondary math endorsement	F	W	26
Sheila	Mech. Eng. Tech	F	Pacific Islander/ W	19
Brian	Lib. Arts	M	W	19
Janet	Pre-med	F	W	20
Greg	DDS	M	W	21
Daniel	Business	M	W	29
Kenny	Physical Therapy	M	W	28
Kathy	Social work	F	W	20
Anthony	Biochem/biotech	M	(no response)	32
Carol	Graduate: math endorsement	F	W	37
Sam	Vet medicine	M	B	22
Kevin	Technology	M	W	28
Thomas	Airways science	M	W	19

Table 6

Summary of Data Collection

Data form	Research questions addressed	Timeframe Week starting							
		6/27	7/4	7/11	7/18	7/25	8/1	8/8	8/15
Student questionnaire (Appendix C)	1	x							
Teacher interviews (Apendix D)	1, 2	x	x					x	x
Artifacts	2	x	x		x	x	x	x	x
Student interviews (Appendix E)	1, 2			x	x			x	
Observations: (Appendix F)	1, 2	x	x	x	x	x	x	x	x

Table 7
Conceptual Categories and Descriptions used in N6

Category	Definition or description
Community	A parent node to subcategories of different types of community and factors related to community such as roles, social norms, and relationships.
Teacher role	Illuminates the nature of the teacher's role in the community and includes subcategories of themes that emerged.
Support	Teacher indicates they will support students in learning.
Listen	Evidence the teacher listens to students.
Mathematical authority	When the teacher determines the correctness of a mathematical answer or statement by a student without giving mathematical reasons, or reason appeals to another authority such as the text.
Intermediate authority	When a teacher indicates that a student answer or comment is correct or incorrect based on a mathematical reason, or when a teacher explains and interprets the mathematics for students.
Student role	Indications of the students' roles in the classroom community.
Listen	Evidence students' roles are to listen.
Active	Further subdivided into two categories: Student initiate: when a student asks a question or makes a statement with the intention of directing the discussion or getting a question answered. Agency: students' own actions to understand the math.
Social norms	Events initiating or maintaining social norms and participants' statements about social norms.
Values	Evidence the community values something. Time became a subcategory since there was evidence that decisions were made based on lack of time and also evidence that there was plenty of time.
Beliefs about math	Statements indicating participants' beliefs about mathematics. The following themes emerged and were used as nodes: Developed by people, real life, procedures, elitism.

Table 7 (continued).

Category	Definition or description
Discussion-based	Evidence of students participating or wanting to participate in discussion during class. Also, includes a subcategory of Spacious: Evidence of opportunities to discuss math with peers and contribute ideas in class.
Beliefs about learning math	Statements indicating participants' beliefs about mathematics. Themes emerged from questionnaire responses and participants' statements and were used to code: examples, play, perseverance, memorizing, conceptual, practice, and discussion.
Affective	Comments and events with respect to feelings about mathematics.
Teacher-student relationships	Questionnaire results about or indications of teacher-student relationships.
Student-student relationships	Evidence of relationships between students in and out of class and questionnaire results of students' values of relationships with peers.
Cases	Three subcategories: Mr. Reilly, Mr. Anderson, and Researcher. All transcripts, fieldnotes, and questionnaires were quick-coded at either Mr. Reilly or Mr. Anderson to facilitate generating reports specific to their cases. Subcategories for each student were created below their respective teachers so reports could be generated on each student.
Researcher	Evidence that my presence affected students' ways of participating, and memos of how the research affected my ways of thinking about my own teaching, or my thoughts on how the research was going.
Interactions	Further subcategorized into four themes: Communication: see row *communication* in this table. Technology: any references to or use of calculators. Sociomathematical norms: see Table 2 for descriptions. Tasks: when students were given a task to work on.
Communication	Subcategories included: Brendefur and Frykholm's (2000) four types of communication: see Table 3 for descriptions.
Questions from students	Further subcategorized into the four themes that emerged: check their thinking, seeking an explanation, expand the discussion, seeking a rule.

Table 7 (continued).

Category	Definition or description.
Explanation	Further subcategorized into two categories of student and teacher explanations so I could examine the nature of the explanations.
Eliciting	“An action intended to ascertain how students interpret the information introduced by the teacher” (Lobato et al., 2005).
Mathematics	The treatment or approach of the mathematics itself as well as the content. Subcategories in rows below.
Relational	Knowing what to do and why (Skemp, 1987).
Instrumental	Knowing procedures without knowing why (Skemp, 1987).
Procedural	Purpose of the discussion is to learn a procedure.
Conceptual	"By conceptual content, we refer to ideas, images, meaning, why a procedure works, one's comprehension of a mathematical situation, and connections among ideas" (Lobato et al.).
Functions	Nature of functions is discussed.
Applications	Applications that are worked in class (whether the teacher, students, or whole-class works on them).
Explore/Play	Purpose of discussion was to explore mathematics.
Common mistakes	Focus of teacher statement is about common mathematical mistakes.
Mathematical conventions	Discussion focused on conventions, such as how notation is used or how angles are named, etc.
Make sense	Participants' references to making sense and mathematical discussions focused on understanding why.
Multiple solutions	Multiple solutions were generated or it was suggested that multiple solutions were possible.
Meaning-making	When the goal of the statement is to convey the meaning of the mathematical object or process or when the communication indicates shared or not shared meanings.

Table 7 (continued).

Category	Definition or description
Multiple representations	More than one representation is used to discuss a problem, concept, or procedure.
Ways of knowing	Subcategorized using previous literature; see Table 1 for descriptions.

Table 8

Questionnaire Responses: During Math Class, What Are Some Things You Do to Help You Learn? Mr. Reilly's Class

Responses	Sample quotes	Number of students
Take notes	"Take good notes, follow practice problems." (Steve)	7
Out of class work	"Lots of out of class work." (Jeremy)	4
Do examples	"Do many examples." (Reggie)	2
Attentive, listen, watch	"Notes, listen, watch board." (Natalie)	2
Follow teacher's methods	"Follow teacher and copy what he does." (Julie)	2
Ask questions	"Being attentive, ask questions, do the assignments. (persistence)" (Mark)	1
Work with a partner	"Take really good notes, have a study partner." (Shawna)	1
No answer	(Nick)	1

Note. Fourteen students responded to the questionnaire. The total number of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 9

Questionnaire Responses: During Math Class, Do You Like to Work With a Partner or In a Group? Why or Why Not? Mr. Reilly's Class

Responses	Sample quotes (see note)	Number of students
Either a partner or a group	"I like to work with one or two other people to help get a good grasp on the math/homework." (Shawna)	5
Partner, but not a group	"Partner, because we equally can contribute and not get lost in a big group." (Karen)	4
Alone	"No, different learning techniques collide." (Jeremy)	4
Alone or with a partner	"I'd rather work alone or with only one other person because I always end up doing the work in a group situation." (Susan)	1

Note. It appeared from their answers that some students thought they were being asked if they preferred working with a partner or with a group. Some referred to homework.

Table 10

Questionnaire Responses: Do You Like to Get to Know Your Teachers And/or Other Students in the Class? Mr. Reilly's Class

Responses	Sample quotes	Number of students
Yes	"Yes helps discussions if comfortable with each other." (Jeremy) Six students wrote "Yes" but did not explain.	8
Yes, teacher, but not necessarily students	"I especially like to know the teacher, the class isn't as important to me." (Karen)	3
Sometimes	"Depends on the class and teacher, math I think is a more solo class." (Ryan)	2
No	(Tim)	1

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 11

Questionnaire Responses: Do You Like It When a Math Teacher Involves the Class in Discussion? Why or Why Not? Mr. Reilly's Class

Responses	Sample quotes	Number of students
Yes	"Yes, helps teach the concepts." (Jake) "Yes but math is not really discussion oriented; right or wrong answers." (Jeremy)	8
No	"No, I'd rather listen to the teacher." (Karen) "No, I am usually tired and don't care. I am not a math major, I just want to finish the requirement with a high grade." (Ryan)	3
Depends	"Yes, and no, it sometimes is helpful, but most time can be confusing." (Julie)	3

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 12

Questionnaire Responses: Do You Usually Offer Input During Class Discussions? Why or Why Not? What Kinds of Input Do You Usually Offer (Ask Questions, Make Suggestions,…)? Mr. Reilly's Class

Responses	Sample quotes	Number of students
Yes, I will *ask questions*	"Yes, I feel if you do not ask you will never learn and I am paying and I want to get the most out of it." (Tim)	9
No	"Not usually, I just compare what I think to what others say." (Nick)	5
Yes, I will *answer questions*	"Yes, I will ask questions, make suggestions, answer problems, so forth." (Susan)	3
Sometimes	"Ask questions if I am really comfortable, otherwise I stay silent." (Julie) "No, I like to listen and take it in. If I have a question, though, I'll ask." (Karen)	2

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 13

Questionnaire Responses: Does Listening to Other Students' Questions or Explanations Help You Learn? Explain. Mr. Reilly's Class

Responses	Sample quotes	Number of students
Yes, no explanation given	"Yes"	5
Yes, their ideas	"Yes; I learn new ideas from other students." (Mark)	4
	"Sometimes it helps if they have a different perspective on it." (Karen)	
Yes, their questions	"Yes, because they usually ask the same questions I have running through my mind." (Susan)	4
Sometimes, but it may confuse me	"Yes and no; if they can explain with pictures I can understand but if they just explain with words I have a hard time visualizing." (Shawna)	2

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 14

Students' Ways of Knowing: Mr. Reilly's Class (Baxter Magolda, 1992; Brew, 2001)

Type	Student	Evidence
Silence		"I don't need to understand where it comes from, I just do it." (Karen) Listening to others may be confusing (Julie).
Absolute	Karen Susan Nick Reggie Owen Jake Steve Jeremy Ryan Mark Julie Natalie (Tim)	Ten responses in Table 16 indicating that the teacher should do examples and show students how to do problems. Five responses in Table 18 refer to memorizing steps and procedures as the way to do mathematics. Eight students in Table 19 referred to practicing and one added listening. Four more said they memorize steps. All of the responses in Table 15 except Owen's answer of "visual." Math is not discussion-oriented, right or wrong answers (Jeremy). All of the responses in Table 8. Mr. Reilly's role as mathematical and intermediate authority. No opportunities for peer interaction during class.
Transitional	(Tim) Shawna	Shawna's statement, "Show different ways on how to do problems." Mr. Reilly's insistence on understanding concepts.
Independent		Natalie discussed the importance of solving a problem on her own in spite of having the teacher's solution nearby (Interview, August 10).
Procedural (Belenky et al., 1997; Brew, 2001)		Mr. Reilly's appeals to students to consider different ways of verifying identities and compare them to determine quality (Fieldnotes, July 12).
Contextual/ Constructive		No evidence.

Note. Students' questionnaire responses were coded, if possible, by the ways of knowing they indicated. Students were placed in the category that most of their statements indicated. Tim is in both absolute and transitional because he had the same number of statements in both categories.

Table 15

Questionnaire Responses: How Do You Best Learn Math? Mr. Reilly's Class

Responses	Sample Quotes	Number of students
Practice, repetition	"By practicing constantly." (Mark)	8
Examples, step-by-step procedures	"Showing through examples step by step." (Susan)	6
Teacher	"Through being taught." (Reggie)	5
Visual	"Visual." (Owen)	1

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 16

Questionnaire Responses: During Math Class, What Are Some Things a Teacher Can Do That Help You Learn? Mr. Reilly's Class

Responses	Sample Quotes	Number of students
Do examples	"Examples, work through completely." (Julie)	10
Involve students	"Get students involved, not just lecture." (Owen)	2
Multiple solutions	"Show different ways on how to do problems." (Shawna)	1
Be clear and prepared	"Be clear and prepared." (Steve)	1
Go slowly	"Go slowly." (Natalie)	1
Create a relaxed classroom environment	"Make the classroom feel easy going without pressure." (Susan)	1

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 17

First Five Class Days: Communication Type by Percent of Text. Mr. Reilly's Class

Date	Unidirectional	Contributive	Reflective	Instructive
June 27	86	14	0	0
June 28	69	31	0	0
June 29	58	42	0	0
June 30	61	39	0	0
July 5	69	31	0	0

Table 18

Questionnaire Responses: Do You Find That Memorizing Steps and Formulas is Important in Mathematics? Explain. Mr. Reilly's Class

Responses	Sample quotes	Number of students
Yes, that is how mathematics is learned	"Yes, it is the most important – because math is rules to follow to complete a problem." (Julie)	5
Yes, for some things but not all	"Yes for formulas because sometimes it is just necessary." (Natalie)	4
No	"No, because I will have my book to look everything up some day." (Tim)	3
Remember through practice	"Not really memorizing the information but using it repeatedly so that it becomes natural." (Susan)	1
Not sure	"Yes/no not sure if I need them." (Owen)	1

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 19

Questionnaire Responses: When Learning New Math Concepts, What Do You Do To Understand the Ideas, Where the Formulas Come From, and Why You Take the Steps You Do? Mr. Reilly's Class

Responses	Sample quotes	Number of students
Practice	"By using them over and over." (Owen)	8
I memorize steps	"I don't need to understand where it comes from, I just do it." (Karen)	4
Listen	"Listen and practice hard (persistence)" (Mark)	1
Understand	"Both, I like to know the concepts and the theory." (Jake)	1
No answer	(Shawna)	1

Note. Fourteen students responded to the questionnaire. The sum of the numbers of students may be more than fourteen since some students' answers could be categorized in more than one category.

Table 20

Questionnaire Responses: How Do You Best Learn Math? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Explanation from someone else	"When a teacher explains everything and doing homework." (Greg)	7
Practice	"Practice and a good instructor." (Sarah)	4
Examples	"Hands on example." (Brian)	4
	"Seeing it done." (Anthony)	
Book	"Show up to class, read the book, then just sit down and do the work!" (Thomas)	1
Visual	"Visual" (Sheila)	1
Group work	"Practice/group work" (Carol)	1
No answer	(Kenny)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 21

Questionnaire Responses: During Math Class, What Are Some Things a Teacher Can Do That Help You Learn? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Explain	"Explain every step." (Greg)	6
Do examples	"Work problems explaining fully what is going on on the board." (Kenny)	5
Give students opportunities to be active	"Give us time to do examples and do multiple ones on the board." (Sarah)	2
Understand the modalities of learning	"Be knowledgeable! Clear and understand the modalities of learning." (Carol)	1
Make class interesting, relaxed	"Explain problems in detail and be funny and cool about it; keep class interesting." (Sam)	1
Write clearly	"Write clearly on the dry erase board." (Anthony)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 22

Questionnaire Responses: Do You Usually Offer Input During Class Discussion? Why or Why Not? What Kinds of Input Do You Usually Offer (Ask Questions, Make Suggestions,…)? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Yes, I'll *ask* questions	"Yes, I ask questions if I do not understand something." (Brian)	6
No	"No, I have a hard time giving input in a math class." (Kathy)	4
Yes, I'll offer other input	"Yes, I will sometimes put input into the classroom." (Thomas)	2
Sometimes	"Sometimes, depending on confidence, questions." (Carol)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 23

Questionnaire Responses: Does Listening to Other Students' Questions or Explanations Help You Learn? Explain. Mr. Anderson's Class

Responses	Sample quotes	Number of students
Yes, their questions	"Listening to their questions and hearing the teacher's explanation helps." (Kathy)	9
Yes, their responses	"Yes, helps if there are different ways of solving." (Kevin)	3
Sometimes	"Sometimes, other times it will just confuse things." (Daniel)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 24

Questionnaire Responses: Do You Like It When a Math Teacher Involves the Class in Discussion? Why or Why Not? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Yes	"Yes, makes you stay alert and focused. Also, someone might say something that you're thinking of." (Sarah) "Yes because it makes you double-think your answer." (Thomas)	11
No	"No, not in math. I like direct lecture with Q/A to follow." (Anthony)	1
Sometimes	"Not really, but sometimes it may be helpful." (Kathy)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 25

Questionnaire Responses During Math Class, What Are Some Things You Do to Help You Learn? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Take notes	"Take notes on how the teacher does his or her work." (Thomas)	6
Practice	"Practice the problems." (Kathy)	4
Interact with others	"Interactive, time to practice." (Carol)	2
Outside of class work	"Homework" (Daniel)	2
Look in book	"Look in book while learning a concept, take notes and ask questions." (Sheila)	1
Sit in front row	"Sit in front row." (Sam)	1
Ask questions	"Look in book while learning a concept, take notes and ask questions." (Sheila)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 26

Students' Ways of Knowing: Mr. Anderson's Class (Baxter-Magolda, 1992; Brew, 2001)

Type	Students	Evidence
Silence		"I don't need to know where it comes from." (Greg Questionnaire, June 27)
Absolute	Sarah Brian Greg Daniel Kenny Kathy Anthony Carol Sam Kevin Thomas	"I just memorize the material, the formulas come from class itself and the books" (Brian Questionnaire, June 27). "take notes on how the teacher does his or her work" (Thomas Questionnaire, June 27). Seven students in this class responded they learned mathematics best by having it explained (see Table 20); ten of the students said the teacher should do examples and/or explain clearly when asked what the teacher can do to help them learn the math (see Table 21). Sarah calls Carol for help since Carol can read solutions from her solution's manual (Sarah Interview, July 13). Mr. Anderson's role as validator.
Transitional	Sheila	Daniel's attempts to understand where some ideas come from (e.g. Fieldnotes, July 27). Mr. Anderson's demonstration of using understanding to explain procedures or solve problems (e.g. Fieldnotes, July 18). Several of Sheila's answers on her questionnaire stressed that she was trying to understand (Sheila Questionnaire, June 27).
Independent	Janet	Janet's answers on the questionnaire indicated she believed she, rather than the instructor, was responsible for her learning (Janet Questionnaire, June 27).
Procedural		No evidence.
Contextual/ Constructive		No evidence.

Table 27

Questionnaire Responses: Do You Find That Memorizing Steps and Formulas Is Important in Mathematics? Explain. Mr. Anderson's Class

Responses	Sample quotes	Number of students
Yes, it is essential to success in mathematics	"Yes, it is impossible to proceed without them" (Brian). "Yes, memorization is important for me to remember the steps in a process" (Anthony).	10
Yes, for some things	"No, formulas can always be looked up. Steps are good to memorize." (Daniel)	2
No	"No, notes help more." (Carol)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 28

Questionnaire Responses: When Learning New Math Concepts, What Do You Do To Understand the Ideas, Where the Formulas Come From, and Why You Take the Steps You Do? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Explanations	"Have someone maybe explain ideas more fully. Because I usually cannot read, then apply I must work problem after taking notes on the idea." (Kevin)	5
Practice	"Practice to perfection." (Janet)	3
Read the book	"Read the book." (Sam)	2
I do not need to understand	"I don't need to know where it comes from." (Greg)	2
Discussion	"Discuss with others, ask instructor." (Carol)	1
No answer	(Kenny)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 29

Questionnaire Responses: During Math Class, Do You Like To Work With a Partner or In a Group? Why or Why Not? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Either partner or group	"Yes, chances are someone will have an explanation of how to do a problem that you can understand." (Sheila)	7
Partner only	"Partner, it is difficult for me to study in a large group." (Brian)	4
Alone	"I like to work alone. I have to take the test alone, so I like to work alone." (Daniel)	3

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 30

Questionnaire Responses: Do You Like to Get to Know Your Teachers And/or Other Students in The Class? Mr. Anderson's Class

Responses	Sample quotes	Number of students
Yes	"Yes, but in a natural way. Not forced by the instructor." (Anthony)	7
Yes, I like to know the teacher	"Students-not really. Teacher – of course, he's who I must really learn from." (Kevin)	3
Sometimes	"Sometimes it just depends." (Thomas)	2
No	"Not really." (Greg)	1

Note. Thirteen students responded to the questionnaire. The sum of the numbers of students may be more than thirteen since some students' answers could be categorized in more than one category.

Table 31

First Five Class Days: Communication Type by Percent of Text. Mr. Anderson's Class

Class period	Uni-directional	Contributive	Reflective	Instructive
June 27	90	10	0	0
June 28	41	59	0	0
June 30	66	34	0	0
July 5	43	57	0	0
July 7	58	42	0	0

Table 32
Framework of Community, Interactions, and Students' Ways of Knowing

Domains	Students	Instructor
Level 1		
Nature of mathematical knowledge	Certain, one right way to think about it. Expresses a need to be told how to do the mathematics	Portrays mathematics as certain and disjoint from real life.
Nature of learning mathematics	Memorize procedures and expect instructor to provide step-by-step directions. Learning requires listening and following examples. Practice to memorize steps and problem types.	Demonstrates procedures without reasons or connections; may not have made sense of the mathematics themselves, and may believe students do not need to make sense. May give lists of steps and non-mathematical ways to remember or to successfully reproduce procedures.
Role of peers/ social norms	No need to listen to other students, it may confuse.	Does the math, evaluates student responses as right or wrong. Answers all questions and does not encourage students to listen to each other.
Relationships	Relationships with peers are not important, or are only important in making the class more comfortable.	No intentional development of teacher-student or student-student relationships. No opportunities provided during class for students to work together.
Interactions related to mathematics/ Communication	Answers with short answers. Responses may be cue-based, indicate pseudo-conceptual behavior, or memorized procedures. Asks questions that indicate they are seeking a rule. Uses technology to calculate but does not reflect on results to see if the answer makes sense. May use drill software to remember procedures and copy examples.	Fosters only unidirectional communication. Closed questions include funneling. Responses to students' incorrect answers may be to tell them the "right" way or remind them of a rule. Takes one student's correct answer as evidence the class understands.
Students' ways of knowing	Indicates silence or absolute knowing.	Maintains absolute ways of knowing, especially received.

Table 32 (*continued)*

Domains	Students	Instructor
Level 2		
Nature of mathematical knowledge	Mostly certain but accepts that instructors' and authors' approaches may differ.	Portrays mathematics as certain, but indicates there may be several ways to solve the same problem.
Nature of learning mathematics	Listen to teacher explanations to increase understanding. May choose to memorize or understand when they practice, depending on the difficulty of the mathematics.	Procedural fluency is the goal, explains more in an effort to help students understand why, however, may permit students to memorize if understanding is difficult.
Role of peers/ social norms	Listen to other students' questions and the instructors' answers. Students do not volunteer explanations and may resist when pressed. Students give answers to the instructor rather than to the class.	Explains students' answers for the rest of the class. Does not press students to explain their reasoning or accepts procedural explanations. Uses only correct student responses to demonstrate the math. Mediates whole-class discussion.
Relationships	Getting to know each other makes the classroom more comfortable. Work in groups when convenient. Group work is valuable for checking answers or sharing knowledge gained from instructor. However, ask instructor to confirm answers.	Provides opportunities for students to work together, but may also provide opportunities for students to avoid working in groups by maintaining their intermediate authority. Provides opportunities for students to know them.
Interactions related to mathematics/ Communication	Asks questions seeking an explanation. Only answer closed questions or contribute solutions when they think they know the right answer, but their answers may not be clear enough for other students to understand. If technology is used, may use it to illustrate multiple representations, but does not reflect on connections or meanings in ways different than suggested by instructor.	Fosters unidirectional and contributive communication by asking students how they solved problems, but does not make connections between solutions. Takes one student's correct answer as evidence the class understands. Provides opportunities for students to practice new techniques in class and receive feedback.
Students' ways of knowing	Indicates absolute or transitional knowing.	Maintains and affirms absolute and transitional knowing.

Table 32 (*continued)*

Domains	Students	Instructor
Level 3		
Nature of mathematical knowledge	Sometimes uncertain, accept that authorities may have different perspectives, recognize there are different solution paths to the same problem, and that mathematics has real life connections.	Portrays mathematical knowledge as useful in real life. Encourages multiple solutions to problems. Suggests there are different approaches and perspectives, but does not encourage students' unique perspectives.
Nature of learning mathematics	Read textbook and listen to teacher to understand but do not critically examine and resolve contradictions that arise. Strive to understand concepts and reasons for steps when they solve problems or practice procedures.	Develops concepts and connects procedures to concepts. Expects students to use understanding to solve problems they have not been shown how to do, but may expect them to use instructor's reasoning.
Role of peers/ social norms	Listen to other students' solutions and share theirs, comparing ideas, but may still rely on teacher to make final pronouncements of correctness.	Uses student contributions and ideas to develop math, but still maintains some intermediate authority. Provides students opportunities to work together.
Relationships	Work in groups to listen to others and share ideas. Gets to know the instructor.	Fosters own relationship with students, offers supportive comments that demonstrate they care about student learning. Provides opportunities for students to work together, but still maintains some intermediate authority.
Interactions related to mathematics/ Communication	Contributes explanations and listens to others, striving to understand and make connections, but still may seek instructor authority to determine correctness. Technology may provide ways to make connections and reduce reliance on instructor's authority.	Fosters contributive communication by asking students how they solved problems. Provides opportunities during class for students to solve problems and receive feedback based on conceptual understanding. May construct an understanding of students' current understanding and offer new problems to help them understand the concepts.
Students' ways of knowing	Indicates transitional knowing.	Confirms and challenges absolute knowing; supports transitional and independent knowing.

Table 32 (*continued)*

Domains	Students	Instructor
Level 4		
Nature of mathematical knowledge	Uncertain. Students appreciate their unique perspective and some solution paths may be better than others depending on the context.	Portrays math as developed by humans throughout history, through discussions with others, and as still being developed and has real life connections. Encourages students to develop their own perspectives.
Nature of learning mathematics	Strive to understand concepts. Understand and flexibly apply procedures. Engages in "doing mathematics" and reflects on those activities. Strive to resolve dissonances.	Develops concepts and connects procedures to concepts. Provides opportunities for students to engage in tasks at a high-level cognitive demand and supports implementation at a high level. Uses student ideas to develop concepts and student thinking to guide instruction.
Role of peers/ social norms	Peers provide ideas to reflect on and discuss. Students engage in mathematical argumentation with each other and determine correctness by the validity of the argument.	Expects students to explain reasoning and uses focusing to encourage students to listen to others. Expects students to determine mathematical validity by argumentation – does not provide final pronouncements of correctness. Fosters the sociomathematical norms in Table 2. Rejects role of authority.
Relationships	Seeks to know peers because peers contribute ideas that can be used to reflect on; students listen to others to challenge their own ideas. Seeks collegial relationship with the instructor.	Fosters relational community by allowing students opportunities to know them, providing support for learning, and providing opportunities and support for student collaboration.
Interaction related to mathematics/ Communication	Communicates their mathematical ideas and critically reflects on others' ideas. If technology is used, it is used to make connections and verify and test conjectures independent of the instructor.	Fosters reflective and instructive communication. If technology is used, uses it to foster communication, conceptual understanding, exploring, refuting or supporting conjectures, making connections between multiple representations.
Students' ways of knowing	Indicates independent and contextual knowing.	Challenges absolute and transitional knowing, and fosters independent and contextual ways of knowing.