PURCHASING, INVENTORY CONTROL, PRICING, AND CONTRACT DESIGN UNDER PURCHASING PRICE UNCERTAINTIES

By

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To the Faculty of Washington State University:

The members of the Committee appointed to examine the dissertation of Xiangling Hu find it satisfactory and recommend that it be accepted.

Chair

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Abstract

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This dissertation focuses on exploring how companies design and adjust purchasing, inventory, and selling strategies when facing stochastic purchasing prices.

First we develops a supply contract that considers environments with changing prices, we then investigate characterization properties of the price processes, and determine expressions of the contract's expected low price and its second moment for a given horizon, then we identify an expected optimum time before the contract expires at which the lowest price occurs. Simulation experiments verify our analysis, and they illustrate how the optimum purchase time decreases as the change rate of the cost increases.

Next, we analyze purchasing strategies for retailers regarding the best timing and amount of purchases when operating under combined timing and quantity flexibility contracts in an environment of uncertain prices. To decrease the computational complexity and make the procedure adaptable to the case of multiple suppliers, we develop, analyze, and compare a Time Strategy and a Target Strategy and then combine these methods into

an approximate algorithm to facilitate the purchasing decision in a more efficient way. We then extend the solution procedure to the multiple suppliers case.

Last we study the problem of planning the procurement and sales for a newsvendor for whom the price of the raw material fluctuates along time and the demand of the output product is random and price-sensitive. After we provide a backward deduction method to solve this problem, we provide an efficient solution algorithm adapted for multiple-supplier cases and long-term-length scenarios, and a corresponding lower bound for the expected profit. We further analyze how to choose between a forward contract and spot market purchasing. Then we extend the above analysis to the profit and risk analysis in multiple-supplier cases and multiple-period newsvendor cases. Through numerical analysis we demonstrate how the potential supplier base and the parameters influence profit and risk, and the purchasing decisions.

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Dedication

This dissertation is dedicated to my husband, my little daughter, my mother and my father.

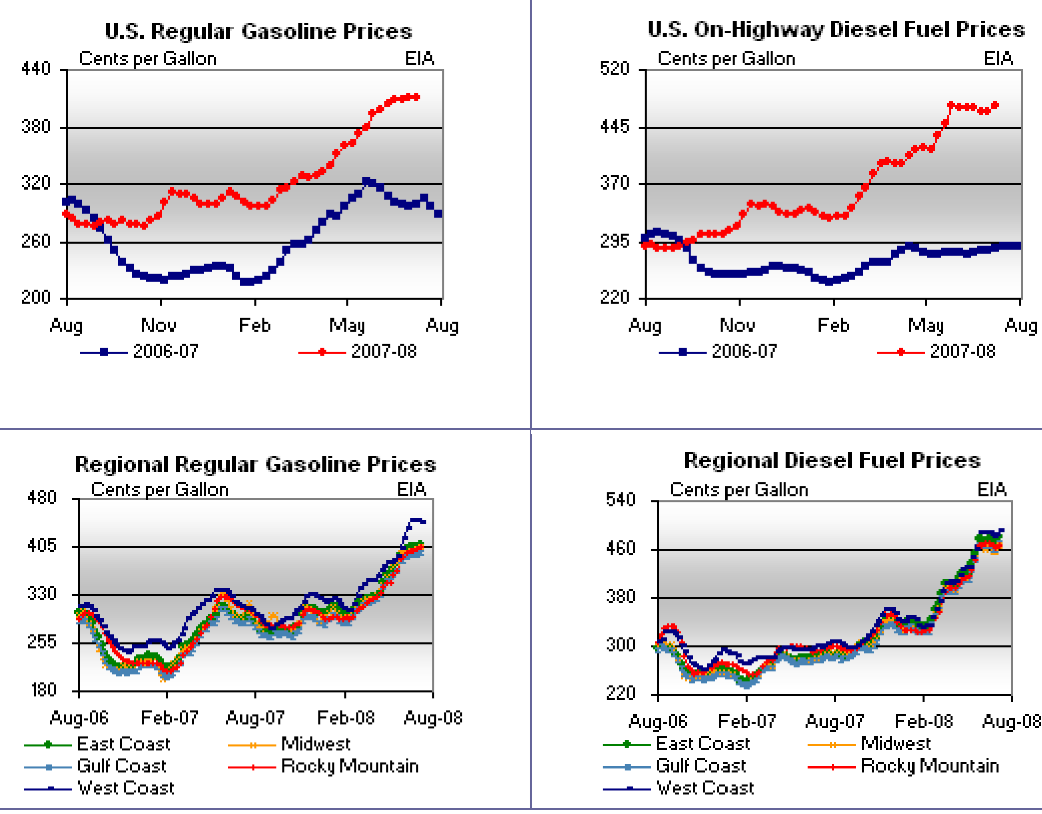
Thank you for your support!

CHAPTER I: BACKGROUND

Price uncertainties are pervasive in many industries, and they can significantly deteriorate profits if left unmanaged. For example (Nagali et al. 2002), the price of DRAM memory used by HP dropped by over 90% in 2001 and more than tripled in 2002. Palladium prices for Ford doubled over the year 2000 and then decreased by over 50% in 2001. Figure 1 lists the gasoline and diesel fuel prices in the past two years. The gasoline price fluctuated a lot and increased from about 290 cents per gallon in August 2006 to about 415 cents per gallon in July 2008 with a positive cost trend. The diesel fuel price also fluctuated a lot and increased from about 295 cents per gallon in August 2006 to about 460 cents per gallon in July 2008 with a positive cost trend. For any firm purchasing commodities such as these, careful purchasing programs must be implemented to try to alleviate the potentially devastating effects of wildly fluctuating prices

Purchase prices fluctuate for a variety of reasons, including exchange rate movements, uncertainty of supply, lack of a futures market, information disclosure, hyperinflation conditions, technical developments, political events, environmental influences, and changing risk preferences of consumers. For example, floating exchange rates may cause a buyer to pay substantially more or less than the original contract price (Carter and Vickery 1988), especially when the contract terms are expressed at an agreed-upon purchase price in the supplier's home currency. Nevertheless, “For the numerous purchasing managers of a global manufacturing concern, the presence of risk-sharing agreements still implies purchasing price uncertainty, even if there exists a contract at an agreed-upon purchase price in the buyer's home currency”(Arcelus et al., 2002). Figure 2 and 3 show the exchange rates from US Dollar to Chinese Yuan and from US Dollar to Canadian Dollar, respectively, from April 4, 2008 to July 4, 2008. In the figures, the exchange rate from US Dollar to Chinese Yuan drops about 22% in these three months, while the trend of exchange rate from US Dollar to Canadian Dollar is flatter. The fluctuations of both exchange rates are severe, especially the exchange rate from US Dollar to Canadian Dollar.

Figure 1. Gasoline and Diesel Fuel Prices in the US



http://tonto.eia.doe.gov/oog/info/gdu/gasdiese1.asp

Figure 2. US Dollar to Chinese Yuan

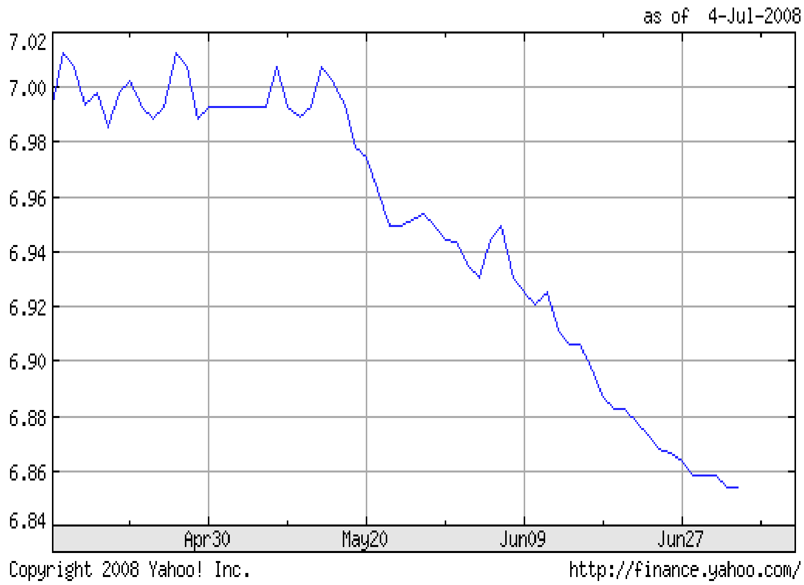
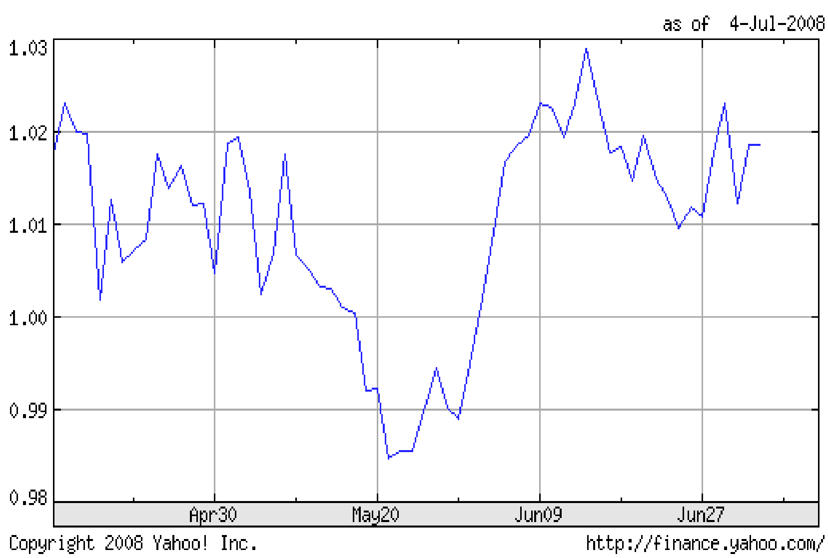


Figure 3. US Dollar to Canadian Dollar



When raw material purchasing prices fluctuate significantly, proper design of a purchasing strategy to hedge risk and increase profits becomes critical. According to Metagroup (2003), lower price ranks highest among the valid reasons driving manufacturers to outsource. Systematic purchase-price risk is the major financial risk to consider in inventory control (Berling and Rosling, 2005). To manage the risk and control the cost, purchasing at an earlier time when the price is relatively low is a common practice in industry. For example, to handle the increase in fuel cost, a lot of the airlines have already purchased futures for those fuels to hedge their prices as time goes on. Containing fuel costs is a key to maintaining profit margins for Southwest Airlines (Cart et. al., 2002). With fuel being an airline's most important variable cost, Southwest's measures have become a model for the industry. In fact, 70% of the fuel needed by Southwest for 2008 is purchased years ago. And Since 1999, hedging has saved Southwest $3.5 billion. Purchasing at an earlier time is also being implemented by the other airline companies (Schreck, 2008). For example, 55% of the fuel used in 2000 by Delta was purchased in 1997 (Cobbs, 2004). However, this earlier purchasing is not always profitable. In 2006, Delta reported a loss of $108 million from the trading when oil prices dropped midyear (Micheline, 2008). Examples can also be found in other industries. Ford posted a $1 billion loss on precious metals inventory and forward contract agreements in December 2001, and Dell’s announcement in October 1999 about the impact of higher-than-expected memory prices resulted in a 7% decline in its stock in one day (Metagroup, 2003). As we can see, purchasing time decisions for raw materials with significant price fluctuations can greatly impact profits. This management dilemma provides the motivation for this dissertation.

In this dissertation, we use Black-Scholes equation to describe underlying purchase price movements. Merton (1973) was the first to publish a paper expanding the mathematical understanding of the options pricing model. Black and Scholes (1973) improved the model and developed "Black-Scholes" options pricing model. The fundamental indication of Black-Scholes is that the option is implicitly priced if the stock is traded. Their research was founded on work developed by scholars such as Louis Bachelier, A. James Boness, Sheen T. Kassouf, Edward O. Thorp, and Paul Samuelson (MacKenzie and Millo 2003, MacKenzie 2003). Merton and Scholes received the 1997 Nobel Prize in Economics for this and related work.

The Black-Scholes model contributes to our understanding of a wide range of contracts with option-like features. For example, the Black-Scholes model explains the prices on European options, which cannot be exercised before the expiration date. The basic idea of the model is that the options are equivalent to a portfolio constructed from the underlying stocks and bonds, and investors gain profits from gaps in asset pricing.

However, many option related expressions cannot be derived directly using Black-Sholes equations, so we need to refer to the other methodologies. The Binomial Options Pricing model approach is a methodology which is able to handle a variety of conditions for which other models cannot easily be applied. Various versions of the Binomial model are widely used by practitioners in the options markets. This is largely because it models the underlying instrument over time, as opposed to at a particular point, and can therefore be readily implemented in a software environment. Although it is slower than the Black-

Scholes model, it is considered more accurate for longer-dated options and options on securities with dividend payments, and it is able to be used in more extensive environments.

For options with several sources of uncertainty or for options with complicated features, Lattice methods like the Binomial Tree method face several difficulties and are not practical. Choosing the most profitable supplier from multiple suppliers to invest is one of them. Monte Carlo option models are generally used in these cases. Monte Carlo simulations are used to analyze financial models by simulating the various sources of uncertainty affecting their value, and then determining their average value over the range of resultant outcomes. Broadie and Glasserman (1996) use Monte Carlo simulation to estimate security price derivatives. Longstaff and Schwartz (2001) implement a least squares approach in a simulation to value American options. Related work includes Boyle (1997), McLeish (2005) and Robert and Casella (2005), and so on. The Monte Carlo method is more advantageous when the dimensions (sources of uncertainty) of the problem increase. Monte Carlo simulation is, however, time-consuming in terms of computation, and it is not used when the Lattice approaches or formulas are sufficient.

In this dissertation, I implement all three methods in the study and develop some formulas and strategies to ease the calculations of those uncertain price related problems without the complex calculations of Monte Carlo simulation or a multiple dimensional Binomial Tree Lattice approach. As the result of the uncertainties of the purchasing cost and as the cost is so crucial to companies' development, I dig into the purchasing strategies, the contract design and options, the selection of suppliers, the joint pricing strategies, and the corresponding profit and risk analysis of firms' purchasing and selling decisions. In the next chapter, I investigate characterization properties of the price processes and then employ these expected price and second moment values to identify an expected optimum time before the contract expires at which to purchase. In Chapter III, I analyze purchasing strategies for retailers regarding the best timing and amount of purchases when operating under combined timing and quantity flexibility contracts. In Chapter IV, I study the problem of planning the procurement and sales for a newsvendor in the case that the newsvendor has a particular time period before the commencement of the selling season to make the purchase and that the demand for the product is random and selling price-sensitive. Finally, I conclude and propose future research in Chapter V.

CHAPTER II: FLEXIBLE SUPPLY CONTRACTS UNDER PRICE UNCERTAINTY

1. Introduction

The purpose of this study is to investigate supply contracts subject to an environment of uncertain prices. I consider the situation where a firm signs a contract with its supplier for the purchase of a certain amount of a material in order to satisfy its customers' future demand. I suppose that a deterministic demand D needed by time T is fixed. Further, I assume that the firm specifies the amount of material needed, but at the same time, I assume that the time of purchasing the material should be flexible within the period $[0,T]$. A "time flexible" contract allows the firm to specify the purchase amount over a given period without specifying the exact time of purchase. Fixed-quantity contracts can arise in numerous settings, for example, purchasing supplies in response to contractual commitments with the buyer's customers, or purchasing supplies to prepare for a fixed six-month production plan, etc. Examples of time-flexible contracts from industry include HP (Nagali et al., 2002) and Ben and Jerry's entry into the Japanese market (Hagen, 1999).

Some recent literature analyzing supply contracts of a specific form include Lee and Namias (1993), Porteus (1990), Tsay et. al. (1999), Bassok and Anupindi (1997), Li and Kouvelis (1999), and Milner and Kouvelis (2005), just to name a few. Our study is closely related to Li and Kouvelis (1999).

In this chapter, and consistent with Dixit and Pindyck (1994), Hull (1997), Li and Kouvelis (1999), Kamrad and Siddique (2004), and Berling and Rosling (2005), I assume that the market material price per unit satisfies the usual Black-Scholes equation. The price is a process $\{S(t), t \geq 0\}$, which can be expressed by the stochastic differential equation

$$\mathrm{d}S(t) = S(t)\{\mu\, \mathrm{d}t + \sigma\, \mathrm{d}W(t)\}, \qquad t \geq 0, \tag{1}$$

where $\mu \in \mathbf{R}$ denotes the usual appreciation rate and $\sigma \in \mathbf{R}^+$ represents the volatility rate. For purely modeling purposes, I assume that both the appreciation and volatility rates are constants. The process $\{W(t), t \geq 0\}$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbf{F} = \{\mathscr{F}_t, t \geq 0\}, P)$, where Ω is a space of continuous functions such that $W(0) = 0$ a.s., $E[W(t)] = 0$ and $E\left[W(t)^2\right] = t$, $t > 0$. The geometric Brownian process $\{S(t), t \geq 0\}$ is by now well known in financial economics and is routinely used to model prices under uncertainty (see, e.g., Karatzas and Shreve, 1988 or Øksendal 1995).

Continuous time models built out of Brownian motion play a crucial role in modern mathematical finance. These models provide the basis of most option pricing, asset allocation and term structure theory currently being used. The examples referred to above have been routinely modeled in the literature, using as a basis model (1). These models imply that the log-returns over intervals of length $\delta > 0$ are normal and independently distributed with a mean of $\left(\mu - \sigma^2/2\right)\delta$ and a standard deviation of σ. Unfortunately, for moderate to large values of δ (corresponding to returns measured over five-minute to one-day intervals), returns are typically heavy tailed, exhibit volatility clustering and are skewed. For higher values of δ, a central limit theorem seems to hold and so Gaussianity becomes a less poor assumption for the log-returns (see, e.g., Campbell et al, 1997). This means that at this "macroscopic" time scale every single assumption underlying the Black-Scholes model is routinely rejected by the type of data usually seen in practice. Given the empirical facts, I strive to improve model (1) by adding a compound Poisson process into (1) (see e.g. Dufresne and Gerber, 1993). This extension coincides with jump diffusion processes and constitutes the family of Lévy processes, i.e., processes expressed as a linear combination of Brownian motion and a pure jump process. Thus, Lévy sample paths are more credible to fit asset prices over time than the traditional standard Brownian motions with drifts.

To sketch how supply contracts under price uncertainty work, I assume that the firm will pay the supplier $S_t > 0$ dollars per unit when purchasing at time *t*. The total number of units needed for the project to complete by time *T* is *D*. The *D* units are not necessarily purchased at the same time. The time flexible contract allows them to be partitioned throughout the period *T*. The purchasing cost of the unit is a function of the spot price at a purchased time *t*. Thus, given a supply contract, the firm's decision is to determine when each purchase occurs and how many units are required to be purchased each time, such that the expected net present value (NPV) of the purchasing cost plus the inventory holding cost is minimized. This is referred as the discounted total cost at time *t*. The purchasing cost of the material and the inventory holding cost are discounted at a fixed annual rate $r > 0$. Thus, the process $\{B(t), t \geq 0\}$ is described by a deterministic differential equation given by

$$\mathrm{d}B(t) = B(t) r \mathrm{d}\, t \text{ or } B(t) = S_0 e^{rt}, \qquad t > 0. \tag{2}$$

If the firm purchases a unit at time *t* and uses it to satisfy the demand at time *T*, then the purchasing cost becomes S_t and the holding cost for the same unit is $S_t(\exp(g(h;T-t)-1))$, where $g(h;T-t)$ is a continuous positive function of the holding coefficient "*h*" and the difference *T*- *t*. Obviously, when *t*=*T*, $g(h;0)=0$, i.e., there is no holding cost. I let $g(\cdot;\cdot)$ be differentiable at its second argument. Thus, the discounted total cost $DTC := \{DTC(t),\ t \geq 0\}$ per unit is expressed by

$$DTC(t) = \frac{S(t)}{B(t)} \exp(g(h;T-t)), \quad t > 0. \tag{3}$$

Using the above formulation, the buyer should decide what instant of time to pay the supplier. It is therefore of interest to study those periods of time that the process DTC spends below certain levels. In particular, conditioning upon DTC being below its initial level, it is of concern to examine the expected duration that the process remains below that level. Ideally, it would be of use to provide table values below the initial process value with their corresponding probability values as a function of time $t \in [0,T]$. Further, knowledge of the unconditional and conditional probabilities related to the expected minimum of the process by time *T* (and even for times earlier than *T*) would play a significant role in the firm's decision making. To that end, I will propose extensive analytical procedures in determining the expected minimum of the process and its second moment by the expiration time and will also compute an expected optimum time within the horizon time that the minimum value of the process has been achieved.

The layout of this chapter is as follows. Section 2 outlines the model of interest. In Section 3 I propose extensive analytical procedures in order to understand how the discounted total cost process behaves throughout the period *T*. Specifically, I determine the expected minimum of the discounted total cost of a unit process and its variance. I then utilize these quantities to obtain an optimum time, prior to the time *T*, such that the discounted total cost process achieves its minimum. In Section 4 I demonstrate via simulations the existence of an optimal stopping time, which varies according to the drift term. Finally, I offer concluding remarks in Section 5.

2. The discounted total cost process

I begin with some preliminary steps. Specifically, in solving equation (1), one simply uses that the price process *S* satisfies

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}, \ \mu \in \mathbf{R}, \ \sigma > 0 \text{ and } t > 0. \tag{4}$$

Using the one-dimensional Ito's formula, one can then have that

$$\mathrm{d}DTC(t) = \left(\frac{\partial DTC(t)}{\partial t} + \mu S(t) \frac{\partial DTC(t)}{\partial S(t)} + \frac{1}{2}\sigma^2 \frac{\partial^2 DTC(t)}{\partial^2 S(t)} \right) \mathrm{d}t + \sigma \frac{\partial DTC(t)}{\partial S(t)} \mathrm{d}W(t),$$

which leads to

$$\mathrm{d}DTC(t) = DTC(t) \left\{ \left(\mu - r - \frac{\sigma^2}{2} \right) \mathrm{d}t - \frac{\partial g(h;T-t)}{\partial t} + \sigma \mathrm{d}W(t) \right\}. \tag{5}$$

The solution of (5) as in (4) can be then shown to satisfy

$$DTC(t) = DTC(0) \exp\left\{ \left(\mu - r - \frac{\sigma^2}{2} \right) t + g(h;T-t) + \sigma W(t) \right\}, \quad \mu, r \in \mathbf{R},\ \sigma > 0 \text{ and } t > 0. \tag{6}$$

It is analytically convenient one to analyze the log-*DTC* price instead of the actual discount total cost process. Thus, using once more the Ito's formula, it can be shown that

$$Y(t) := \log DCT(t) = Y(0) + \left(\mu - r - \sigma^2/2 \right) t + g(h;T-t) + \sigma W(t),\ \mu \in \mathbf{R},\ \sigma > 0 \text{ and } t > 0. \tag{7}$$

Assuming that g is purely linear (see e.g., Li and Kouvelis, 1999), i.e., $g(h;T-t)=h(T-t)$, equation (7) is then formed as

$$Y(t)=Y(0)+hT+\left(\mu-r-h-\sigma^2/2\right)t+\sigma W(t)=\sigma\{u+\theta\, t+W(t)\},\tag{8}$$

where $u:=Y(0)^{1/\sigma}+\dfrac{hT}{\sigma}$ and $\theta=\dfrac{\mu-r-h-\sigma^2/2}{\sigma}$ with $\mu, r, h \in \mathbf{R}^+$ and $\sigma>0$.

It is known that $\dfrac{\mu-r}{\sigma}$ represents the market price of risk. Further, transforming (8), the process of interest then becomes $X(t):=Y(t)/\sigma=u+\theta\, t+W(t)$, with $W(t)$, $t>0$, being the standard Brownian motion shifted at u (u>0) , and with drift θ.

3. Development of the optimum rule

In this section I begin by preparing a few elementary inequalities of the distribution of the log-*DTC*. We establish the distribution of the minimum of the log-*DTC*, which is further utilized to obtain the first two moments. Finally, we demonstrate the existence of an optimum stopping time, and we provide a mechanism to compute the expected optimum time.

Using the Kolmogorov's forward equation, it is not hard to see that the diffusion equation (8) satisfies

$$\frac{\partial}{\partial t} f = -\theta \frac{\partial}{\partial y} f + \frac{\partial^2}{2\partial y^2} f \,, \tag{9}$$

where $f(t, y|s, x) = \frac{\partial}{\partial y} P\big(X(t) \le \mathrm{y} \,|\, X(s) = x \big), \theta = \frac{\mu - r - h - \sigma^2/2}{\sigma}$ and $x \in \mathbf{R}$ and $0 \le s < t$. The transition probability distribution from $X(t)$ to $X(t+1)$ $t \ge 0$, satisfies normal distribution.

In light of (9), we observe that the following elementary inequalities for the process $X(t), t \ge 0$ are satisfied.

Lemma 2.1. *The process* $X = \{X(t), t \ge 0\}$ *satisfies*

$$P(X(t) \le u) \begin{cases} < \frac{1}{2} & \theta > 0 \\ = \frac{1}{2} & \theta = 0 \\ > \frac{1}{2} & \theta < 0 \end{cases} .$$

Note that as *t* increases, the probability that the process will stay below level *u* by time *t* tends to zero, remains equal to $\frac{1}{2}$, or tends to one according to θ being greater than, equal to, or smaller than zero, respectively. To understand the speed at which the probability $P(X(t) \le u), \theta > 0$, tends to zero as a function of the time *t,* the following inequality is of great help:

$$\frac{\theta\sqrt{t}}{1+\theta^2 t}\varphi\left(\theta\sqrt{t}\right) < P\left(X(t)\leq u\right) < \frac{1}{\theta\sqrt{t}}\varphi\left(\theta\sqrt{t}\right), \quad t\in[0,T]. \tag{10}$$

When $\theta < 0$, the inequalities in (10) are reversed. Along the same vein, one can also show that the process X drifts towards ∞, oscillates or drifts towards $-\infty$, as θ is positive, zero or negative, respectively. For the case of $\theta > 0$, it is clear that the process X will stay below the level u (if it stays) for only a finite amount of time. Eventually, it will cross the u level and will finally drift towards infinity. Similar arguments also hold if $\theta < 0$ but the statements above are now reversed. It is, however, clear that the case of $\theta < 0$ cannot occur in most contract scenarios.

To develop the following property, let $I(T) := \inf_{t\leq T} X(t)$ denote the minimum of the process $X(t), t\geq 0$ by time T. Incorporating Dassios (1995), it can be shown that for $\theta\in\mathbf{R}$ and $T\geq 0$, the minimum $I(T)$ is absolutely continuous, i.e., $P(I(T)\in \mathrm{d}x) = f_T(x)\mathrm{d}x$ (using the reflection principle) with probability density function given by

$$f_T(x) = \begin{cases} \left(\frac{2}{\pi T}\right)^{1/2}\exp\left(-\frac{(x-u-\theta T)^2}{2T}\right) + 2\theta\exp\left(2\theta(x-u)\right)\Phi\left(\frac{x-u+\theta T}{\sqrt{T}}\right), & x<u \\ 0, & x\geq u, \end{cases} \tag{11}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\mathrm{d}y$, and $x\in\mathbf{R}$.

Substituting $\theta = 0$ into (11), one can then demonstrate that

$$P(I(T) \in \mathrm{d}x) = \begin{cases} \left(\frac{2}{\pi T}\right) \exp\left(-\frac{(x-u)^2}{2T}\right) \mathrm{d}x, & x < u \\ 0, & x \geq u\,. \end{cases} \tag{12}$$

The density of log-*DTC* in (12) represents the case where the process oscillates around its initial level.

In view of (11), Theorem 2.1 below establishes computational expressions for the first two moments of the random variable $I(T)$.

Theorem 2.1. *The first two moments of* $I(T)$*, the standardized minimum of log-DTC process by time* $T \geq 0$*, are given by*

$$E[I(\mathrm{T})] = \left(u - \frac{1}{2\theta}\right)\Phi\left(\theta\sqrt{T}\right) + \left[\sqrt{\frac{T}{2\pi}} - \frac{1}{2}\left(u - \frac{1}{2\theta}\right)\right]\exp\left(-\theta^2 t/2\right) + 2(u + \theta T)\Phi\left(-\theta\sqrt{T}\right),$$

$$E\left[I^2(\mathrm{T})\right] = \left[u^2 - \frac{7u}{2\theta} + \frac{1}{2\theta^2}\right]\Phi\left(\theta\sqrt{T}\right) + \left\{\frac{T\sqrt{2}}{2} - 2(u + \theta T)^2\right\}\Phi\left(-\theta\sqrt{T}\right)$$

$$+\left\{\frac{5\theta T^{3/2}}{\sqrt{2\pi}} - \frac{T}{2} + 2\left(u + \frac{1}{\theta}\right)\frac{\sqrt{T}}{\sqrt{2\pi}} - \frac{1}{2}\left\{\left(u - \frac{1}{\theta}\right)\left(u - \frac{1}{2\theta}\right) + 1\right\}\right\}e^{-\theta^2 T/2},$$

where $u = Y(0)^{1/\sigma} + \frac{hT}{\sigma}$ *and* $\theta = \frac{\mu - r - h - \sigma^2/2}{\sigma} \in \mathbf{R}$.

Furthermore, as $T \to \infty$, *then*

$$\lim_{T\to\infty} E[I(\mathrm{T})] = \left(u - \frac{1}{2\theta}\right)$$

and for $u < (10\theta)^{-1}$,

$$\lim_{T\to\infty} Var(I(\mathrm{T})) = -\frac{5u}{2\theta} + \frac{1}{4\theta^2}.$$

Proof.

Calling upon (11), we redefine the density of the random variable $I(T)$ as $f_T(x) = (I_1(x) + 2\theta I_2(x))I(x<u)$, where $I(x \in A)$ denotes the indicator function, which becomes one if the event occurs and zero otherwise.

Consequently, $E[I(T)] = \int_{-\infty}^{u} xI_1(x)\mathrm{d}x + 2\theta\int_{-\infty}^{u} xI_2(x)\mathrm{d}x \equiv I_1 + 2\theta I_2$. We thus compute the two parts separately. The first can be expressed as

$$I_1 = 2\left(\frac{T}{2\pi}\right)^{1/2} \exp\left(-\theta^2 T/2\right) + 2(u+\theta T)\int_{-\infty}^{-\theta\sqrt{T}} \varphi(u)\mathrm{d}x$$

$$= 2\left(\frac{T}{2\pi}\right)^{1/2} \exp\left(-\theta^2 T/2\right) + 2(u+\theta T)\Phi\left(-\theta\sqrt{T}\right),\ \theta \in \mathbf{R}, \tag{13}$$

where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\varphi(y)\,\mathrm{d}y$ and $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $x\in\mathbf{R}$.

To determine the second term, we first set $y=x-u$. Further, based upon $2\theta y\exp(2\theta y)\mathrm{d}y=\ \mathrm{d}(y\exp(2\theta y))-\exp(2\theta y)\mathrm{d}y$ and an integration by parts, it follows that

$$2\theta I_2=\int_{-\infty}^{0}\Phi\left(\frac{y+\theta T}{\sqrt{T}}\right)\mathrm{d}(y\exp(2y\theta))+2\theta\left(u-\frac{1}{2\theta}\right)\int_{-\infty}^{0}\exp(2\theta\,y)\Phi\left(\frac{y+\theta\,T}{\sqrt{T}}\right)\mathrm{d}y$$

$$=-\int_{-\infty}^{0}y\exp(2y\theta)\frac{1}{\sqrt{2\pi T}}e^{-(y+\theta T)^2/2T}\,\mathrm{d}y+2\theta\left(u-\frac{1}{2\theta}\right)\int_{-\infty}^{0}\exp(2\theta\,y)\Phi\left(\frac{y+\theta\,T}{\sqrt{T}}\right)\mathrm{d}y$$

$$=-\left\{\frac{\sqrt{T}}{\sqrt{2\pi}}+\frac{1}{2}\left(u-\frac{1}{2\theta}\right)\right\}e^{-\theta^2T/2}+\left(u-\frac{1}{2\theta}\right)\Phi\left(\theta\sqrt{T}\right). \tag{14}$$

Using (13) and (14), the expected value of $I(T)$, $T\geq 0$, immediately follows.

Arguing the same way as in the determination of the first moment, we have that $E\left[I^2(T)\right]=\ \int_{-\infty}^{u}x^2I_1(x)\mathrm{d}x+2\theta\int_{-\infty}^{u}x^2I_2(x)\mathrm{d}x\equiv\widetilde{I}_1+2\theta\widetilde{I}_2$. The first term $\widetilde{I}_1$ can be expressed as

$$\widetilde{I}_1=T\frac{\sqrt{2}}{\sqrt{2\pi}}\int_{-\infty}^{-\theta\sqrt{T}}v^2e^{-v^2/2}\mathrm{d}v+4(u+\theta T)\frac{\sqrt{T}}{\sqrt{2\pi}}\int_{-\infty}^{-\theta\sqrt{T}}ve^{-v^2/2}\mathrm{d}v\;-2(u+\theta T)^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\theta\sqrt{T}}e^{-v^2/2}\mathrm{d}v$$

$$=T\frac{\sqrt{2}}{\sqrt{2\pi}}\int_{\theta^2T/2}^{\infty}v^{1/2}e^{-v}\mathrm{d}v+4(u+\theta T)\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\theta^2T/2}\;-2(u+\theta T)^2\,\Phi\left(-\theta\sqrt{T}\right). \tag{15}$$

Since $\Gamma(a+1,x)=a\Gamma(a,x)+x^ae^{-x}$ and $\Gamma\left(1/2,x^2/2\right)=\sqrt{2\pi}\left(1-\Phi(x)\right)$, equation (15) can be expressed as

$$\widetilde{I}_1 = \left(\frac{T\sqrt{2}}{2} - 2(u+\theta T)^2 \right) \Phi\left(-\theta\sqrt{T}\right) + \frac{(4u+5\theta T)\sqrt{T}}{\sqrt{2\pi}} e^{-\theta^2 T/2}.$$

To obtain the second term, we again set $y = x - u$ and then it follows that

$$2\theta\widetilde{I}_2 = 2\theta \int_{-\infty}^{0} y^2 \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y + 2u \left\{ 2\theta \int_{-\infty}^{0} y \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y \right\}$$

$$+ u^2 \left\{ 2\theta \int_{-\infty}^{0} \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y \right\}. \qquad (16)$$

Since $2\theta \int_{-\infty}^{0} y \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y = 2\theta I_2 - u 2\theta \int_{-\infty}^{0} \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y$,equation (16) can be replaced by

$$2\theta\widetilde{I}_2 = 2\theta \int_{-\infty}^{0} y^2 \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y$$

$$+ 2u 2\theta I_2 - u^2 2\theta \int_{-\infty}^{0} \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y \qquad (17)$$

Using the fact that $2\theta y^2 \exp(2\theta y)\mathrm{d}y = \mathrm{d}\left(y^2 \exp(2\theta y)\right) - 2y \exp(2\theta y)\mathrm{d}y$, the first term in (16) can be further simplified by

$$2\theta \int_{-\infty}^{0} y^2 \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y = \int_{-\infty}^{0} \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}\left(y^2 \exp(2\theta\, y)\right) - 2\int_{-\infty}^{0} y \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y$$

$$= -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{0} y^2 \exp(2\theta\, y) e^{-(y+\theta\, T)^2/2T}\, \mathrm{d}y - I_2 - u \int_{-\infty}^{0} \exp(2\theta\, y) \Phi\left(\frac{y+\theta\, T}{\sqrt{T}} \right) \mathrm{d}y. \qquad (18)$$

Substituting (17) into (18), we obtain

$$2\theta\,\widetilde{I}_2 = -\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{0} y^2 \exp(2\theta\, y) e^{-(y+\theta\, T)^2/2T}\, \mathrm{d}y$$

$$+2\left(u-\frac{1}{\theta}\right)(2\theta I_2) - u\left(u+\frac{1}{2\theta}\right)\left\{2\theta\int_{-\infty}^{0} \exp(2\theta\, y)\Phi\left(\frac{y+\theta\, T}{\sqrt{T}}\right)\mathrm{d}y\right\}. \qquad (19)$$

Note that the first term $\widetilde{I}_{21}$ in (19) can be written as

$$\widetilde{I}_{21} = -\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{0} y^2 \exp(2\theta y)\, e^{-(y+\theta T)^2/2T}\mathrm{d}y = -\frac{T\sqrt{2}\Gamma(3/2)}{\sqrt{2\pi}} e^{-\theta^2 T/2} = -\frac{T}{2} e^{-\theta^2 T/2}. \qquad (20)$$

Further, it is easy to see that

$$2\theta\int_{-\infty}^{0} \exp(2\theta\, y)\Phi\left(\frac{y+\theta\, T}{\sqrt{T}}\right)\mathrm{d}y = \Phi\left(\theta\sqrt{T}\right) - \frac{1}{2} e^{-\theta^2 T/2}. \qquad (21)$$

In light of (13), (20) and (21), it then follows that

$$2\theta\widetilde{I}_2 = \left\{2\left(u-\frac{1}{\theta}\right)\left(u-\frac{1}{2\theta}\right) - u\left(u+\frac{1}{2\theta}\right)\right\}\Phi\left(\theta\sqrt{T}\right)$$

$$-\left\{\frac{T}{2} + 2\left(u-\frac{1}{\theta}\right)\frac{\sqrt{T}}{\sqrt{2\pi}} + \frac{1}{2}\left\{\left(u-\frac{1}{\theta}\right)\left(u-\frac{1}{2\theta}\right)+1\right\}\right\} e^{-\theta^2 T/2}. \qquad (22)$$

Combining (16) and (22), the computation of the second moment is now completed.

The second part of the theorem easily follows by just letting $T \to \infty$.

The following result presents a stochastic integral representation of the infimum of the log-*DCT*. It will play a significant role in identifying an optimum time before the horizon *T,* at which the loss function becomes minimum. Specifically, if $\mathscr{M}$ denotes a family of Markov times with respect to $\sigma-$algebra $\mathscr{F}$, then $\tau^* \in \mathscr{M}$ is chosen such that

$$E\left[\left(I(T)-X\left(\tau^*\right)\right)^2\right]=\inf_{\tau\in\mathscr{M}} E\left[\left(I(T)-X(\tau)\right)^2\right]. \tag{23}$$

This part of the work was influenced by Graversen et al. (2001) and Graversen et al. (2007), who studied the existence of optimum times when the deviation considered in (23) is just the $\max_{0<t\le T} W(t)-W(\tau)$, where $W(\cdot)$ here is only a standard Brownian motion. Even though there are a few similar ideas between the two studies, the investigation here assumes initial values and, of course, assumes Brownian motions with drifts. To that aspect the calculations become more complex and the use of Theorem 2.1 is essential. Further, the aim here is to achieve the ultimate minimum (not the maximum) of the log-*DCT*, which was by no means approached earlier.

Proposition 2.1. *Let* $X(t)=u+\theta\, t+W(t)=u+W_\theta(t),\ t\ge 0$ *denote the log–DTC process. The ultimate minimum of* $X(t)$ *and* $W_\theta(t),\ t\ge 0$ are $I(T)=\inf_{0\le u\le T} X_u$ *and* $I_\theta(T)=\inf_{0\le u\le T} W_\theta(u)$, $T\ge 0$*, respectively. Let* $F_{t-s}(x)=P(I_\theta(t-s)\le x)$ *and* $\mathscr{F}_s=\sigma(X(u):u\le s)$. *Then,*

$$I(T) = E[I(T)] + \int_0^T F_{T-s}(I(s) - X(s))\,\mathrm{d}W(s).$$

Proof. Using the time homogeneity property, it follows that

$$\begin{aligned}
E[I(T)|\mathscr{F}_t] &= I(t) + E[\inf_{t\le s\le T} X(s) - I(t)|\mathscr{F}_t] \\
&= I(t) + E[\{\inf_{t\le s\le T}(X(s) - X(t)) - (I(t) - X(t))\}|\mathscr{F}_t] \\
&= I(t) + E[(I_\theta(T-t) - (I(t) - X(t)))^-] \ ,
\end{aligned} \tag{24}$$

where $x^- = (-x)\vee 0$. It is known that $E[(X-c)^-] = \int_{-\infty}^{c} P(X<z)\,\mathrm{d}z$. Upon substituting the last identity into (24), we obtain that

$$E[I(T)|\mathscr{F}_t] = I(t) + \int_{-\infty}^{I(t)-X(t)} F_{T-t}(z)\,\mathrm{d}z = f(t, X(t), I(t)). \tag{25}$$

Applying Itô's formula to the right hand side of (25) and using the fact that the left hand side defines a continuous martingale, we have

$$E[I(T)|\mathscr{F}_t] = E[I(T)] + \int_0^t \frac{\partial f}{\partial x}(s, X(s), I(s))\,\mathrm{d}W(s)$$

$$= E[I(T)] + \int_0^t F_{T-s}(I(s) - X(s))\,\mathrm{d}W(s). \tag{26}$$

This is a nontrivial continuous martingale and does not have paths of bounded variation.

Setting $t = T$ and then equalizing the left hand sides of (25) and (26) the desired result easily follows.

Using Dassios (1995), equation (21), or Shiryaev et al (1993), it can be seen that

$$F_t(x) = \Phi\left(\frac{-x + \theta\, t}{\sqrt{t}}\right) - e^{2\theta x}\Phi\left(\frac{x + \theta\, t}{\sqrt{t}}\right). \tag{27}$$

Equation (27) will be used for Proposition 2.1 in order to evaluate the cumulative distribution $F_{T-t}(\cdot)$.

The drive of the next result is to find $\{\mathscr{F}_t\}-$ stopping time, $\tau^* \le T$, such that $X(\tau^*)$ is the closest to $I(T) = \inf_{0 \le t \le T} X(t)$ in some sense. Clearly, $X(\cdot)$ describes the evolution of the log-*DTC* process on the interval $0 \le t \le T$. The financial motivation of such a problem is to observe the log-*DTC* and then pay off the contract at its lowest price. The next theorem suggests that such a time does exist. The determination of evaluating the optimum time is very complex and tedious. It is, however, available for the case when the underlying process is only a standard Brownian motion and when the process attains its maximum (see, e.g., Urusov, 2005). Below, we provide various steps of how this can be achieved for the case of Brownian motion with a drift with initial value and when the process attains its minimum.

In light of Proposition 2.1, the main theorem of this chapter is then formulated as follows.

Theorem 2. 2. *Let $\mathscr{M}$ denote a family of Markov times with respect to $\sigma-$algebra $\mathscr{F}$. There exists a stopping time $\tau^* \in \mathscr{M}$, $\tau^* \leq T$, such that*

$$E\left[\left(I(T)-X\left(\tau^*\right)\right)^2\right]=\inf_{\tau\in\mathscr{M}} E\left[\left(I(T)-X(\tau)\right)^2\right].$$

The expression of $E\left[\left(I(T)-X(\tau)\right)^2\right]$ in terms of u, θ, $\tau \in \mathscr{M}$, and $I(T)$ is given by

$$E\left[\left(I(T)-X(\tau)\right)^2\right] = E\left[\int_0^\tau\left\{\frac{\theta^2}{2}s+2\theta\left(u-E[I(T)]\right)+1-2F_{T-s}\left(I(s)-X(s)\right)\right\}\mathrm{d}s\right]$$

$$+uE[I(T)]+u^2 +E\left[I^2(T)\right].$$

Proof. Note that for any $\tau \in \mathscr{M}$, we have

$$E\left[\left(I(T)-X(\tau)\right)^2\right] = E\left[I^2(T)\right]+E\left[X^2(\tau)\right]-2E\left[I(T)X(\tau)\right]. \tag{28}$$

The terms of interest in equation (28) are, of course, the second and the third term. The first term is given in Theorem 2.1. To evaluate the third term, we utilize Proposition 2.1. Specifically, we substitute $I(T)$ by its integral representation, as follows

$$E[I(T)X(\tau)] = E[I(T)]E[X(\tau)] + E\left[X(\tau)\int_0^T F_{T-s}(I(s)-X(s))\,\mathrm{d}W(s)\right]$$

$$= E[I(T)]E[X(\tau)] + E\left[X(\tau)\left\{\int_0^\tau F_{T-s}(I(s)-X(s))\,\mathrm{d}W(s) + \int_\tau^T F_{T-s}(I(s)-X(s))\,\mathrm{d}W(s)\right\}\right]. \quad (29)$$

Note that $\int_\tau^T F_{T-s}(I(s)-X(s))\,\mathrm{d}W(s)$ and $X(\tau)$ are independent, thus the second product in the expectation (29) vanishes. Since $X(\tau) = u + \theta\tau + \int_0^\tau \mathrm{d}W(s)$, one can then use the Itô's isometry property to finally obtain

$$E[I(T)X(\tau)] = E[I(T)](u + \theta\,E[\tau]) + E\left[\int_0^\tau F_{T-s}(I(s)-X(s))\,\mathrm{d}s\right]. \quad (30)$$

To evaluate the second term in (28), we note that

$$E\left[X^2(\tau)\right] = E\left[(u + \theta\,\tau + W(\tau))^2\right] = u^2 + (2u\theta + 1)E[\tau] + \theta^2 E\left[\tau^2\right]. \quad (31)$$

Combining (28), (30) and (31), we obtain

$$E\left[(I(T)-X(\tau))^2\right] = \theta^2 E\left[\tau^2\right] + \{2\theta(u - E[I(T)]) + 1\}E[\tau] - 2E\left[\int_0^\tau F_{T-s}(I(s)-X(s))\,\mathrm{d}s\right]$$

$$+uE[I(T)] + u^2 + E\left[I^2(T)\right]. \qquad (32)$$

This completes the proof of Theorem 2.2.

To complete our investigation, as equation (32) remains complex, we seek to obtain a more explicit expression. To achieve more thorough understanding of (3.4), one can condition the loss function and notice that $E\left[(I(T)-X(\tau))^2\right] = E\left[E\left[(I(T)-X(\tau))^2\right] \| \tau\right]$. It is thus appropriate to study the inside expectation first. In light of this, the following corollary is in order.

Corollary 2.1. *Let $\tau^* \in \mathscr{M}$ be as in Theorem 2.1. Then the optimum time can be computed from the following:*

$$E\left[\left(I(T)-X\left(\tau^*\right)\right)^2\right] = \inf_{\tau\in\mathscr{M}} E\left[\int_0^\tau \left\{\frac{\theta^2}{2}s + 2\theta(u - E[I(T)]) + 1 - 2\int_{-\infty}^0 F_{T-s}(y)\,\mathrm{d}F_s(y)\right\}\mathrm{d}s\right]$$

$$+uE[I(T)] + u^2 + E\left[I^2(T)\right].$$

Proof. From Theorem 2.2, it is cleat that $E\left[(I(T)-X(\tau))^2\right]$ will change only through the first expectation in the right-hand side. In particular, the change will occur through the term

$$V(\theta)= E\left[\int_0^\tau F_{T-s}(I(s)-X(s))\,\mathrm{d}s\right].$$

It is known (Bertoin, 1996, p156) that the reflected process $\{I(t)-X(t): 0 \le t \le T\}$ is a Markov process in the filtration $\mathscr{F}_t = \sigma(W(u): u \le t)$ and its semi-group has the Feller property. That is, we need only to consider stopping times, which are hitting times for $I(\cdot)-X(\cdot)$.

Using the homogeneity property for the Brownian motions, it can be seen that for any $x < 0$

$$\begin{aligned} P(I(t)-X(t)\le x \mid W(t)=y) &= P(W(s)-W(t)-\theta(s-t)\le x \text{ for some } s<t \mid W(t)=y) \\ &= P(W(t-s)-\theta(t-s)\ge -x \text{ for some } s<t \mid W(t)=y) \\ &= P(W(s)-\theta s\ge -x \text{ for some } s<t \mid W(t)=y). \end{aligned} \tag{33}$$

In view of the well-known formula of Siegmund (1986), we have that

$$P\left(W(s)-\theta s \geq -x \text{ for some } s<t \mid W(t)=y\right)=\exp\left(2x(\theta t-x-y)/t\right), \ x<0. \quad (34)$$

In conjunction with (32) and since $V(\theta)=E[V(\theta \mid \tau)]$, the conditioning upon τ, $V(\theta \mid \tau)$ can be then expressed as follows

$$V(\theta \mid \tau)=\int_0^{\tau} E\left[F_{T-s}(I(s)-X(s))\,\mathrm{d}s\right]$$

$$=\int_0^{\tau} E\left[\Phi\left(\frac{-I(s)+X(s)+\theta(T-s)}{\sqrt{T-s}}\right)-e^{2\theta(I(s)-X(s))}\Phi\left(\frac{I(s)-X(s)+\theta(T-s)}{\sqrt{T-s}}\right)\right]\mathrm{d}s$$

$$=\int_0^{\tau}\int_{-\infty}^{0}\left\{\Phi\left(\frac{-y+\theta(T-s)}{\sqrt{T-s}}\right)-e^{2\theta y}\Phi\left(\frac{y+\theta(T-s)}{\sqrt{T-s}}\right)\right\}\mathrm{d}P\left(I(s)-X(s)\leq y\right)\mathrm{d}s. \quad (35)$$

Using (34) and (35), it follows that

$$P\left(I(t)-X(t)\leq x\right)=1-\int_{-\infty}^{-x+\theta t} P\left(W(s)\leq -x+\theta s \text{ for some } s<t \mid W(t)=y\right)P\left(W(t)\in \mathrm{d}y\right)$$

$$=1-\int_{-\infty}^{-\theta u-y}\left\{1-\exp\left(\frac{2x(\theta t-y-z)}{t}\right)\right\}\varphi\left(\frac{z}{\sqrt{t}}\right)\mathrm{d}z$$

$$=1-\Phi\left(\frac{-y+\theta t}{\sqrt{t}}\right)-e^{2\theta y}\Phi\left(\frac{y+\theta t}{\sqrt{t}}\right)$$

$$=1-F_t(x), \ x<0, \quad (36)$$

and its density is given by (21), for u=0.

Substituting (36) into (35), the result follows immediately.

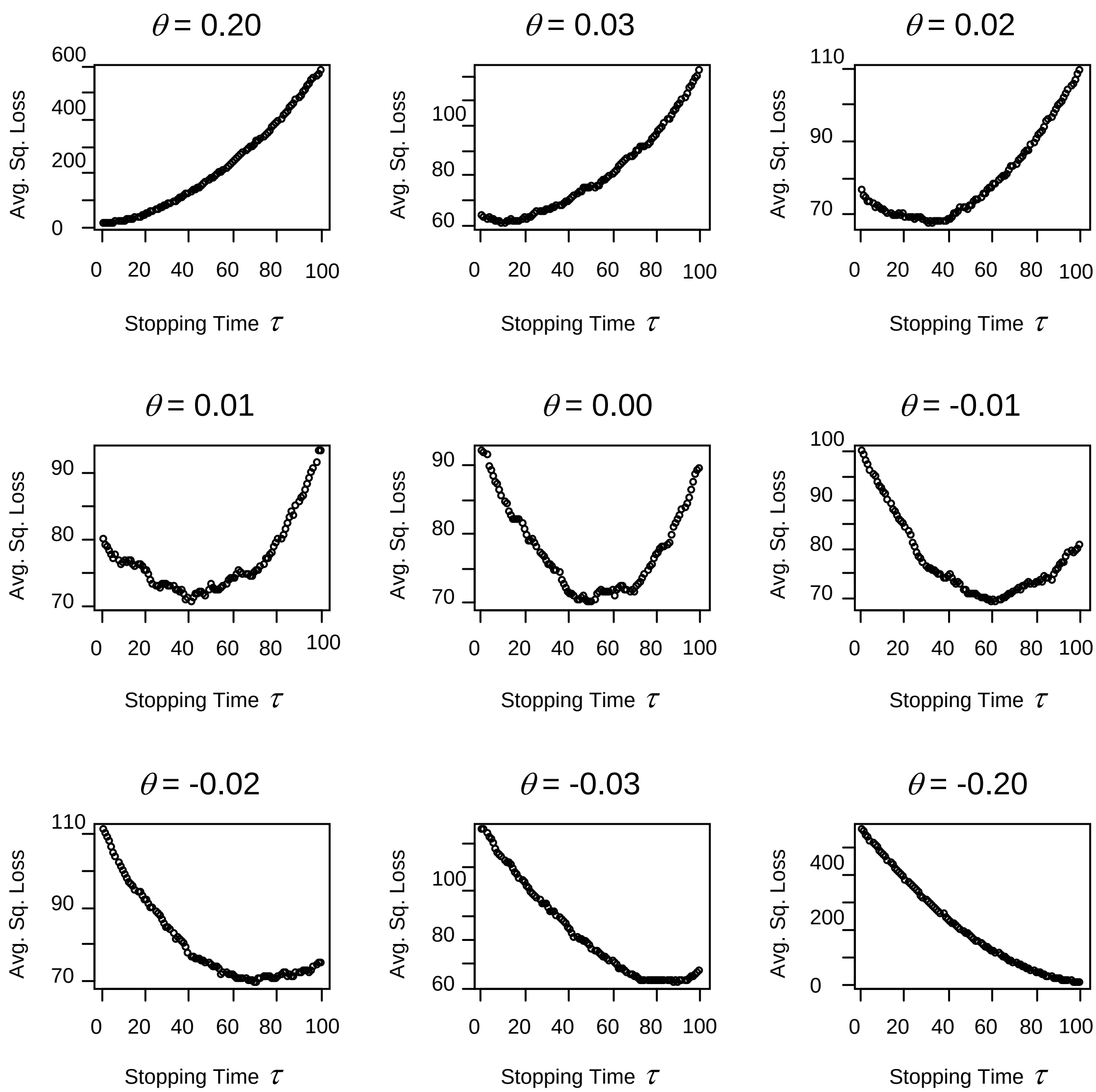
4. Simulation Analysis of Stopping Times

The best stopping time for a given contract is highly dependent upon the drift term θ. When θ is large enough, the units should be purchased at the beginning of the time horizon, and *vice-versa*. However, when the drift term approaches 0, the best purchase time to minimize expected cost will not be at the beginning or the end.

We conducted simulations to illustrate the optimal purchase time under various values of θ. In particular, we generated 2000 price processes of length $T = 100$ periods. We let $\sigma = 1$ and $u = 0$ (note that the results are independent of the value of u chosen). For purchasing at each stopping time period τ between 0 and 100, we calculated the squared loss of the difference between the process $X(\tau)$ and $I(T)$ (the minimum of the process $X(t)$ over the full time horizon). Thus, each graph in Figure 1 represents a plot of 101 points.

Figure 4 presents simulation results for θ values of 0.20, 0.03, 0.02, 0.01, 0.00, -0.01, -0.02, -0.03, and -0.20. The corresponding expected optimal stopping times based on the simulation results were τ = 0, 11, 30, 40, 50, 60, 70, 88, and 100, respectively.

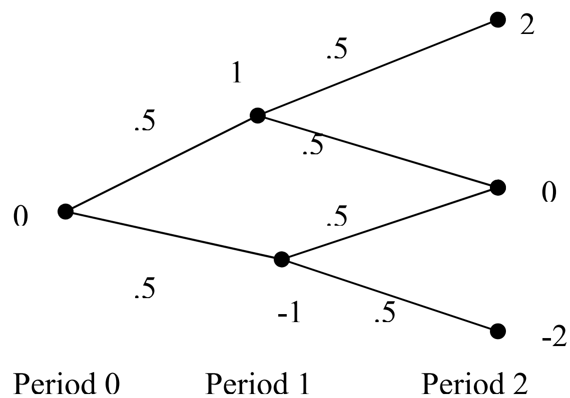
Figure 4. Simulation Results Plotting the Squared Loss Average



From the plots, we notice that when there is a steep upward cost trend, the optimal purchasing time is at the very beginning; when there is a steep downward cost trend, the optimal purchasing time is at the very end; and when there is a level cost tend, the optimal purchasing time is likely to appear in the middle of the time horizon.

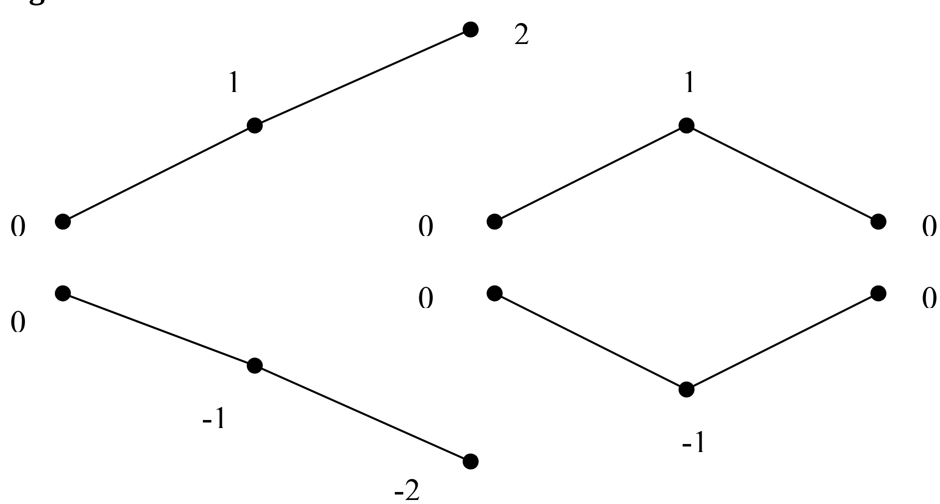
We use a binomial tree to approximate the decision process to further demonstrate our result for the case when the price trend is level. Here is a simple example. The following is a binomial tree with three periods: period 0, period1, and period 2. Each X at period τ satisfies $\begin{cases} X(\tau)_{up} = X(\tau-1)+1 & prob=.5 \\ X(\tau)_{down} = X(\tau-1)-1 & prob=.5 \end{cases}$. The plot is as follows:

Figure 5: An Example of Three Periods Binomial Tree



So there are four paths as shown below, and each path has a probability of .25 to occur.

Figure 6. ALL Possible Paths for the Three Periods Binomial Tree



For these four paths, we can calculate $\left[(I(T)-X(\tau))^2\right]$ for purchasing time $\tau = 0$, $\tau = 1$, $\tau = 2$ and get the following table. Here $T = 2$.

Table 1. Squared Losses for Each Cost Path in the Binormal Tree

Path	Probability	The lowest cost $I(2)$ on that path	$\left[(I(2)-X(0))^2\right]$ at time 0	$\left[(I(2)-X(1))^2\right]$ at time 1	$\left[(I(2)-X(2))^2\right]$ at time 2
0 — 1 — 2	1/4	0	0	1	2^2
0 — 1 — 0	1/4	0	0	1	0
0 — -1 — -2	1/4	-2	2^2	1	0
0 — -1 — 0	1/4	-1	1	0	1
Average of $\left[(I(T)-X(\tau))^2\right]$			**5/4**	**3/4**	**5/4**

From the table, we find that the lowest average square lose $E\left[(I(T)-X(\tau))^2\right]$ is ¾ and it occurs at time $\tau = 1$. It is interesting that the average lose $E(I(T)-X(\tau))$ is all ¾ for time $\tau = 0$, $\tau = 1$, and $\tau = 2$.

The example and our intuition suggest the following explanation for a level cost trend. In terms of choosing a time to attain the lowest cost, any time during the horizon is equally likely to be correct. However, choosing a time in the middle of the horizon reduces the risk of a very poor guess (as represented by the squared loss). In other words, purchasing in the middle of the horizon prevents us from being too far away from the lowest possible cost. Finally, for cost processes with slight increasing (decreasing) cost trends, the optimum purchasing time is between 0 (T) and the midpoint of the horizon to represent a balance between the squared loss minimization phenomenon of a level cost trend and the obvious desire to purchase at the beginning (end) of the horizon for a significant cost trend, i.e. to represent a risk-return tradeoff.

In this section, we have used Monte Carlo simulation to find the optimal purchasing time when the expected square loss is minimum. The corresponding simulation process is as follows： 1) Get the initial values (refer to (1) to (7) for the definitions) , calculate u and θ using $u = Y(0)^{1/\sigma} + \frac{hT}{\sigma}$ and $\theta = \frac{\mu - r - h - \sigma^2/2}{\sigma}$, and choose the simulation length. 2) Input the initial values to the program in the appendix and run the program. 3) Retrieve the results from the outputs of the program. The output of the program will provide the optimal purchasing time and the corresponding plot for the expected square loss against the purchasing time.

5. Concluding Remarks

Today's increased globalization has opened up many more possibilities for procuring goods from around the world. However, the increased options generate even more exposure to purchase price fluctuations, particularly with regard to issues such as exchange rate movements, political turmoil, and supply and demand shifts in emerging markets like China. In response, supply contracts have begun to include a time flexibility component, allowing the buyer to choose the time of purchase. With a risky option of purchase timing in hand, firms need assistance to help determine when, in fact, the best time to purchase might be. This chapter provides a solution for that decision.

In the formulation developed in this chapter we regard log-*DTC* as a state of a game at time *T*, where each realization, $\omega \in \Omega$, corresponds to one sample of the game. For each time period prior to the end of the horizon T, the buyer has the option of stopping the game and accepting the current cost or continuing the game in the hope that purchasing later will reduce the cost further. The problem is of course that we do not know in what state the game will be in the future, we can only estimate the probability distribution of the 'future'. Among all possible stopping times $\tau < T$ in the above formulation, we have demonstrated a procedure for obtaining an optimal time $\tau^* < T$ such that $\log DTC(\cdot)$ give us the best result "in a long run', i.e., the expected loss $E\left[(I(T) - X(\tau))^2\right]$ becomes minimum.

CHAPTER III: PURCHASING DECISIONS UNDER STOCHASTIC PRICES AND TIMING AND QUANTITY OPTIONS

1. Introduction & Literature Review

In response to purchasing and selling uncertainties, researchers and practitioners like IBM, Hewlett Packard, Sun, Compaq, and Solectron have incorporated into purchasing contracts various combinations of quantity and timing flexibility. For example, IBM provides a flexibility option, in which, within the flexibility zone, both due dates and lot quantities can be changed from one procurement plan to the next (Connors, 1995). This contract represents the type of *combined timing and quantity flexibility contract* that we analyze in this chapter. In combined timing and quantity flexibility contracts, the contract specifies neither the exact purchase time nor the exact quantity. Quantity flexibility allows the buyer to purchase an amount within a pre-specified range. Such contracts typically define an α $(0 \le \alpha \le 1)$ quantity flexibility, so the buyer agrees to purchase a maximum of Q units from a supplier, but the buyer can purchase a total of x units, where $(1-\alpha)Q \le x \le Q$. Alternatively, timing flexibility allows the buyer to choose when to purchase within a pre-specified time period. We use the term *fully dynamic contract* to denote a contract that combines timing with quantity flexibility and $\alpha = 1$. The fully dynamic case allows complete quantity and purchase timing freedom for the buyer. Combined timing and quantity flexibility contracts can be thought of as purchasing $(1-\alpha)Q$ from the supplier using a timing flexibility contract without quantity flexibility, while purchasing αQ from the supplier using a fully dynamic contract.

As a consequence of price fluctuations in spot markets, a pure fixed-price contract rarely exists. If the actual work or spot price varies from estimates, the client will often pay the difference (Metagroup, 2003). This is especially true for high-volume commodities. For example, Southwest Gas (William *et al*., 1992) received a proposal from gas producers in the form of a combined timing and quantity flexibility contract in which contract provisions included a commodity rate tied to an index of fluctuating but increasing spot market prices. Even though Carter and Vickery (1988) reported that more than 50% of the surveyed firms used some form of risk-sharing agreements with their suppliers, "For the numerous purchasing managers of a global manufacturing concern, the presence of risk-sharing agreements still implies purchasing price uncertainty, even if there exists a contract at an agreed-upon purchase price in the buyer's home currency." (Arcelus *et al*., 2002). In this chapter, the purchase price at the execution date is assumed to have already taken into account any risk-sharing discount, i.e., the purchase price has already been adjusted by supressing the natural price variability.

Previous research about quantity flexibility contracts with or without timing flexibility has focused on the use of contractual quantity flexibility to handle uncertain demand situations. Sethi *et al.* (2004) discusses single- and multi-period quantity flexibility contracts involving one demand forecast update in each period. Chen and Song (2001) consider nonstationary demand in a multi-echelon setting and introduce a state-dependent, echelon base-stock policy. Tsay (1999) analyzes the impact of system flexibility on inventory characteristics and the patterns by which forecast and order variability propagate along the supply chain. Fotopoulos *et al.* (2006) develop an equation to study the probability of purchasing at a certain cost with timing flexibility contracts, and they determine the optimal time length of a contract. Milner and Kouvelis (2005) consider three demand processes: a standard demand case, a Bayesian demand case, and a Martingale demand case. They analyze how product demand characteristics affect the strategic value of two complementary forms of flexibility: quantity flexibility in production and timing flexibility in scheduling.

Another relevant research stream analyzes how stochastic purchase price affects inventory policies. These papers analyze how to find the best purchasing time under time-flexibility cases. Bjerksund *et al.* (1990) discuss project values and operational decision rules by interpreting investment as an option and the output price as an underlying asset. Li and Kouvelis (1999) develop the optimal purchasing strategies for both time-flexible and time-inflexible contracts with risk–sharing features in environments of price uncertainty. They expand the analysis to two-supplier sourcing environments and quantity flexibility in such contracts. Berling and Rosling (2005) study how to adjust (R, Q) inventory policies under stochastic demand and purchase costs. Li and Kouvelis (1999) analyze fully dynamic contracts for two suppliers, where the total order quantity for these suppliers is fixed and the problem is how to distribute that quantity between them.

In this chapter, we seek to identify the best timing and order quantity to purchase from single or multiple suppliers when operating under purchase price uncertainty. This chapter has three primary contributions. First, the chapter is among the first to address the issue of combining timing and quantity flexibility under price uncertainty. Second, we design a very efficient approximate optimal solution procedure, which quickly decreases the computational complexity to solve the computational problem of multiple suppliers and long time length. Third, we analyze how to select suppliers for different purposes: (1) minimizing downside risk, (2) maximizing profit, and (3) maximizing profit subject to a constraint on downside risk.

The rest of the chapter is organized as follows. In the next section we present the basic profit model. Section 3 studies the optimal purchasing strategies when purchasing from a single supplier. We introduce a Time Strategy and a Target Strategy for both timing flexibility contracts and fully dynamic contracts, and we determine how to combine these methods to produce the best procedures to apply to combined time and quantity flexible contracts. We further present results of a computational study using the solution procedures. Section 4 extends the results to estimate the profit and risk when multiple suppliers are available. Section 5 extends the results to a more general case in which the market price for the final product price is influenced by the price of the raw material purchasing and studies how to invest when multiple correlated suppliers are available. We conclude in Section 6 with some final observations.

2. Modeling Basics

To introduce the model, we assume (like Bering and Mosling, 2005 and Li and Kouvelis, 1999) that the purchase price per unit satisfies the usual Black-Scholes equation

(Black, Fischer and Myron S. Scholes, 1973), i.e., the price at time t is a process $P = \{P(t), t \geq 0\}$, which is expressed by the stochastic differential equation

$$\mathrm{d}P(t) = P(t)\{\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\}, \qquad t \geq 0, \tag{1}$$

where $\mu \in \mathbf{R}$ denotes the average rate and $\sigma \in \mathbf{R}^+$ represents the volatility rate. The process $W = \{W(t), t \geq 0\}$ is the standard Brownian motion satisfying $W(0) = 0$ a.s., $E[W(t)] = 0$ and $E\left[W(t)^2\right] = t$, $t > 0$. The process $P = \{P(t), t \geq 0\}$ is commonly used in modeling uncertain prices (see, e.g., Karatzas and Shreve, 1988).

We model a company obtaining a single product over a time period $[0,T]$ from one or more suppliers, transforming the product into a finished good and selling to the market. Specifically, we assume that the firm will pay the supplier $P(t) > 0$ dollars per unit when the unit is purchased at time *t*. If the firm purchases one unit at time *t* and uses it to satisfy the demand at time T, the holding cost for this unit is $P(t)[h(T-t)]$, where h is the periodic holding cost percentage. To aid us in subsequent derivations, we will use $P(t)[e^{h(T-t)} - 1]$ to approximate this holding cost, which is similar to the Berling and Rosling (2005) approach. Higher-order terms of this Taylor approximation quickly move toward zero when the holding cost is much smaller than P. Next, the cost is further discounted at a constant periodic interest rate $r > 0$. This discount rate represents the buyer's opportunity cost of capital (Li and Kouvelis 1999, Berling and Rosling 2005) and is an important part of the cost because some contracts can extend for more than 20 years (William et al 1992). Then the discounted total cost $C = \{C(t),\ t \geq 0\}$ per unit is expressed by

$$C(t) = e^{-rt}[P(t) + P(t)(e^{h(T-t)} - 1)] \\ = P(t)e^{hT-(h+r)t} \tag{2}$$

From (1), it is easy to derive that the price process P satisfies

$$P(t) = P(0)\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t) \right), \mu \in \mathbf{R},\ \sigma > 0 \text{ and } t > 0. \tag{3}$$

Using the one-dimensional Ito's formula, we then conclude that

$$\mathrm{d}C(t) = \left(\frac{\partial C(t)}{\partial t} + \mu C(t)\frac{\partial C(t)}{\partial p(t)} + \frac{1}{2}\sigma^2 \frac{\partial^2 C(t)}{\partial^2 p(t)} \right)\mathrm{d}t + \sigma \frac{\partial C(t)}{\partial \mathrm{p}(t)}\mathrm{d}W(t) \tag{4}$$

which after some simple calculations, one may obtain that

$$C(t) = p(0)\exp\left\{ hT + \left(\mu - r - h - \frac{\sigma^2}{2} \right)t + \sigma W(t) \right\}, \tag{5}$$

$\mu, r \in \mathbf{R}$, $\sigma > 0$ and $t > 0$.

Subsequently, we shall focus on analyzing the $\log(C(t))$ instead of the actual discounted total cost process. Since $\log(C(t))$ is continuous and twice differentiable on $[0,T] \times \mathbf{R}$, it implies that $Y(t) = \log(C(t))$ is again an Ito process, and

$$Y(t) = \log(P(0)) + hT + \left(\mu - r - h - \frac{\sigma^2}{2} \right)t + \sigma W(t) = \sigma\{u + \theta\, t + W(t)\}, \tag{6}$$

where $u = \dfrac{\log(P(0)) + hT}{\sigma}$ and $\theta = \dfrac{\mu - r - h - 0.5\sigma^2}{\sigma}$ with $\mu, r, h \in \mathbf{R}^+$ and $\sigma > 0$. Based on the above analysis, our strategies rely on how the behavior of the process $w_\theta(t) := Y(t)/\sigma - u = \theta\, t + W(t)$ evolves. Here we define $w_\theta(t)$ as the "discounted price," which follows the Brownian motion at time zero with drift θ. So the discounted total cost $C(t)$ can be expressed as $e^{(u+w_\theta(t))\sigma}$.

We define R as the selling price. In combined timing and quantity flexibility contracts, the buyer must purchase $(1-\alpha)Q$ to satisfy its contracted amount no matter what the price is, and then it has the option to purchase the remaining αQ. The buyer will purchase the remaining αQ at time t only if $R - C(t) > 0$. We use $\prod(t)$ to represent the corresponding buyer's unit profit at time t. Then the firm's objective in timing flexibility cases is $\max_{t\in[0,T]} \Pi(t) = \max_{t\in[0,T]} \left[R - C(t)\right]$. The firm's objective in fully dynamic cases is $\max_{t\in[0,T]} \Pi(t) = \max_{t\in[0,T]} [R - C(t)]^+$. Here $[x]^+ = \max[x,0]$.

3. Purchasing Strategies Using One Supplier

We could employ the Binomial Lattice method to approximate the process of $P(t)$ in the way suggested by Cox *et al*. (1979) and then we design a backward deduction heuristic to solve purchasing problems for timing and quantity flexibility contracts.

Using backward deduction method, the time interval $[0,T]$ is divided into n small time intervals. $\Delta = T/n$. From section 2, we can get total cost at t small interval $C(t)$ satisfy $\mathrm{d}C(t) = C(t)\{(\theta\sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\}$, $W = \{W(t), t \geq 0\}$ is the standard Brownian motion satisfying $W(0) = 0$ a.s., $E[W(t)] = 0$ and $E\left[W(t)^2\right] = t$, $t > 0$. If $C_\theta(t+1)$ represents the approximation of $C(t)$ using Binomial Lattice method introduced by Cox et al (1979), we can get:

$$C_\theta(t+1)=\begin{cases} C_\theta^{up}(t+1)=C_\theta(t)e^{\sqrt{\Delta}\sigma} & prob \quad p=\frac{1}{2}\left(1+(\theta-.5\sigma)\sqrt{\Delta}\right) \\ C_\theta^{down}(t+1)=C_\theta(t)e^{-\sqrt{\Delta}\sigma} & prob \quad q=\frac{1}{2}\left(1-(\theta-.5\sigma)\sqrt{\Delta}\right) \end{cases}$$

The maximized possible expected profit using fully dynamic contract is calculated recursively backward as:

$$U_t(C_\theta(t))=\max\{R-e^{(u+w_\theta(t))\sigma}, pU_{t+1}[C_\theta^{up}(t+1)]+qU_{t+1}[C_\theta^{down}(t+1)],0\}$$

Then the decision function is as follows:

$$\begin{cases} Purchase\ D & if \quad U_t(C_\theta(t))=R-C_\theta(t) \\ Waite & if \quad U_t(C_\theta(t))=pU_{t+1}[C_\theta^{up}(t+1)]+qU_{t+1}[C_\theta^{down}(t+1)] \\ Give\ Up & if \quad U_t(C_\theta(t))=0 \end{cases}$$

Then we can prove that if $\theta > -\frac{\sigma}{2}$, the decision is to purchase at point $C_\theta^{down}(t+1)$, then we should purchase at $C_\theta(t)$.

$$\begin{aligned} U_t(C_\theta(t)) &= \max\{R-e^{(u+w_\theta(t))\sigma}, pU_{t+1}[C_\theta^{up}(t+1)]+qU_{t+1}C_\theta^{down}(t+1)],0\} \\ &= \max\{R-e^{(u+w_\theta(t))\sigma}, pU_{t+1}[C_\theta^{up}(t+1)]+q[R-e^{(u+w_\theta^{down}(t+1))\sigma}],0\} \\ &\le \max\{R-e^{(u+w_\theta(t))\sigma}, p[R-e^{(u+w_\theta^{up}(t+1))\sigma}]+q[R-e^{(u+w_\theta^{down}(t+1))\sigma}],0\} \\ &= R-e^{(u+w_\theta(t))\sigma} \end{aligned}$$

So we know that purchasing should be at the highest point of binomial tree at a certain time period t.

Then we can get the idea to solve this problem is to find out the corresponding level for each time point so that we will purchase at that level.

However, reliable calculations for this heuristic method require each time period to be segmented into many small intervals, which significantly increases the computational burden for a large time horizon scenario. For one supplier and a short horizon, we can use this method to solve the problem, but when there are multiple suppliers, the number of calculations of all the combinations will evolve exponentially. For example, if $T = 20$ and there are 6 suppliers, the computational complexity is about $1000^6 = 10^{18}$. In this chapter, we study the properties of the model and the Brownian motion to develop a fast solution procedure which significantly decreases the computational complexity, especially for the multiple-supplier case. For $\theta < -\sigma / 2$, i.e., when the discounted total cost is trending down, we can even provide a one-step exact solution. Using the above example, the computational complexity of our procedure is approximately 1200.

Next we introduce strategies used in our solution procedure. We will then apply those for two specific cases: the fully dynamic case and the timing flexibility case. Finally, we merge our strategies to explore the combined timing and quantity flexibility case.

3.1. Solution Strategies

We analyze three specific strategies as defined below.

(1) Time Strategy—At the beginning of the time horizon, calculate the expected profit for purchasing in each of the periods in the time horizon. Choose to purchase during the period that corresponding to the highest expected profit.

(2) Target Strategy— At the beginning of the time horizon, calculate the expected profit for purchasing in each of the periods in the time horizon. The highest expected profit becomes the target. As time passes, purchase as soon as purchasing in that period would generate a profit that equals or exceeds the target.

(3) Dynamic Target Strategy—At each new period, as long as the units have not yet been purchased, recalculate the expected profit for purchasing in each of the remaining periods in the time horizon. The highest expected profit calculated becomes the new target. During the next period, purchase if doing so would equal or exceed the new target. If not, then recalculate expected profits, updating the target level to be used in the succeeding period. Continue in this manner until the units have been purchased or the end of the time horizon has been reached.

The Time Strategy attempts to predict today the best purchase time in the future. On the other hand, the Target Strategy focuses on estimating the highest level that expected profit will reach. The purchasing strategy in that case is akin to a "limit order" placed by a stock market investor, wherein the investor directs the broker to sell (or buy) the stock as soon as the stock price reaches a certain "reservation price." Finally, the Dynamic Target Strategy is like the Target Strategy, except that the reservation price (in this case the target unit profit level) is adjusted each period as new information comes in.

3.2. Fully Dynamic Case

Since the buying firm has the option not to purchase any units when using fully dynamic contracts, our strategies incorporate that option. Specifically, if all expected profits are negative when using the Time Strategy, then the firm will choose not to purchase. And for the two target strategies, if the buyer reaches the end of the horizon having not purchased yet, it will only purchase if the profit at that point would be positive.

3.2.1. Time Strategy under the Fully Dynamic Case

Using the Time Strategy, the buying company will purchase D units at time $t^* = \arg\max_{t \in [0,T]} E\left[\Pi(t)^+\right] = \arg\max_{t \in [0,T]} \{R - E\exp[(u + w_\theta(t))\sigma]\}^+$ if $R - \exp[(u + w_\theta(t^*))\sigma] > 0$,

otherwise, the company will not purchase.

Since $w_\theta(t) = \theta t + w(t)$, the corresponding expected value of $\Pi(t)^+$ is as follows.

Proposition 3.1. *In fully dynamic cases, the expected profit of the Time Strategy at time* $t \in (0, T]$ *is*

$$E\,\Pi(t)^+ = R\Phi\left(\frac{\frac{\ln(R)}{\sigma} - u - \theta t}{\sqrt{t}}\right) - \exp(u\sigma + \theta t \sigma + \frac{t\sigma^2}{2})\Phi\left(\frac{\frac{\ln(R)}{\sigma} - u - \theta t - \sigma t}{\sqrt{t}}\right). \qquad (8)$$

Proof:

Let $\log(R) - (u + \theta t)\sigma = v$, then it becomes $\exp(u\sigma + \theta t \sigma)\{\exp(v) - \exp[w(t)\sigma]\}^+$

$$\begin{aligned}
&\exp(u\sigma + \theta t\sigma)\{\exp(v) - \exp[w(t)\sigma]\}^+ \\
&= \int_{-\infty}^{v/\sigma} \exp(u\sigma + \theta t\sigma)[\exp(v) - \exp(x\sigma)]\left[-\frac{\exp(-x^2/(2t))}{\sqrt{2\pi t}}\right]dx \\
&= \int_{-\infty}^{v/\sigma} [R - \exp(u\sigma + \theta t\sigma + x\sigma)]\left[-\frac{\exp(-x^2/(2t))}{\sqrt{2\pi t}}\right]dx \\
&= R\Phi(\frac{v}{\sigma\sqrt{t}}) - \exp(u\sigma + \theta t\sigma + \frac{t\sigma^2}{2})\Phi\left(\frac{v/\sigma - t\sigma}{\sqrt{t}}\right)
\end{aligned}$$

3.2.2. Target Strategy under the Fully Dynamic Case

To further our analysis, we cite a theorem from Sakhanenko (2005):

Theorem 2.1. *Let* $W(t)$ *be the standard Wiener Process defined for* $t \in [0, \infty)$.

Set $\overline{W}(b,T) = \max_{0 \le t \le T}(W(t) + bt)$ *and* $W_\theta(0) = 0$, *then the cumulative distribution of* $\overline{W}(b,T)$ *is*

$$P\{\overline{W}(b,T)\geq x\}=\overline{\Phi}(\frac{x-bt}{\sqrt{t}})+e^{2\theta x}\overline{\Phi}(\frac{x+bt}{\sqrt{t}}),$$

where $\overline{\Phi}(x)=\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$, , *and* $x\geq 0$.

Then deriving from $P(\min_{0\le u\le t} w_\theta(t)\le x)=P(-\min_{0\le u\le t} w_\theta(t)\ge -x)=P(-\max_{0\le u\le t} w_{-\theta}(t)\ge -x)$, we can get the following result.

Corollary 3.1. *The cumulative distribution of* $M_\theta(t):=\min_{0\le u\le t} w_\theta(u)$ *and* $\tau_\theta(x):=\inf\{t\ge 0: w_\theta(t)=x\}$, *where* $w_\theta(t)=\theta t+w(t)$, *is given by*

$P(M_\theta(t)\le x)=P(\tau_\theta(x)\le t)=\Phi\left(\frac{x-\theta t}{\sqrt{\mathrm{t}}}\right)+e^{2\theta x}\Phi\left(\frac{x+\theta t}{\sqrt{\mathrm{t}}}\right)$, $\theta\in\mathbf{R}$, $x<0$, *and* $t\ge 0$, *where*

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}\mathrm{d}y,\ \textit{and}\ x\le 0. \tag{9}$$

This corollary provides us with the probability that the minimum value of the discounted total cost process $w_\theta(t)$ during $[0,t]$ is less than x, i.e., the probability that the target level x has been reached during $[0,t]$. We use this to estimate the expected profit of the Target Strategy.

In choosing the best target cost x to utilize in the Target Strategy, we need to strike a balance between selecting a low x, which would, if reached, provide a low cost, versus selecting a large enough x such that the probability of actually reaching that cost during the time horizon is reasonable. Next we describe a procedure for choosing a good value for x.

We use $\Pi_T(x)$ to represent the profit of the Target Strategy at cost level x. For any target unit profit L, $R - e^{(u+x)\sigma} = L$, so $x = \frac{(R-L)}{\sigma} - u$. Then the problem becomes determining whether the cost level x is ever reached by $w_\theta(t) = \theta t + w(t)$. Furthermore, since $w_\theta(t)$ includes the holding cost h and interest rate r, the purchasing price level y can be computed by $y = x + r + h$ and the problem reduces to comparing the fluctuating price with y to decide whether to purchase right away or wait. Here

$$\begin{aligned}\Pi_T(x) = \{[R - (\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T)]^+ I(\tau_\theta(x) \le T) \\ + [R - (\exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T)]^+ I(\tau_\theta(x) > T)\}.\end{aligned} \tag{10}$$

$$\begin{aligned}E[\Pi_T(x)] = \{E[R - \exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T]^+ P(\tau_\theta(x) \le T) \\ + E[R - \exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T]^+ P(\tau_\theta(x) > T)\}.\end{aligned} \tag{11}$$

Note in (11), when $\theta \ge -\sigma/2$, referring to (7), the second term of (11) satisfies

$$\begin{aligned}&E\{R - [\exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T]\}^+ \\ &\le \{R - E[\exp(u + w_\theta(T))\sigma]\}^+ \\ &= [R - \exp(u\sigma + \theta T\sigma + T\sigma^2/2)]^+,\end{aligned} \tag{12}$$

while the first term of (11) satisfies

$$\begin{aligned}&E[R - \exp(u + w_\theta(\tau_\theta(x))\sigma \mid \tau_\theta(x) \le T]^+ \\ &= [R - \exp((u + x)\sigma)]^+, \text{ where the target level } x \le 0.\end{aligned} \tag{13}$$

When $\theta \ge -\sigma/2$, (13) is larger than (12). Furthermore, if $\theta > 0$, we can show that $\lim_{T\to\infty}(\tau_\theta(x) > T) = 1 - e^{2\theta x}$, and if $\theta \le 0, \lim_{T\to\infty} P(\tau_\theta(x) \le T) = 0$. Thus, when θ is small or less than zero, the second term of (11) goes to zero and is comparatively much smaller than the first term. Moreover, as the value of θ grows, $E[R - \exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T]^+$ is much larger than $E[R - \exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T]^+$, especially for large *T*. Therefore, we can conclude that for any $\theta \geq -\sigma/2$, we can approximate (11) by ignoring the second term. Thus

$$\max_x E[\Pi_T(x)] \approx \max_x [R - E(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \leq T)]^+ P(\tau_\theta(x) \leq T)$$

$$\approx \max_x [R - \exp(u + x)\sigma]^+ P(w_\theta(x) \leq T) \text{ for } \theta > -\frac{\sigma}{2}. \tag{14}$$

Combining (9) and (14), we obtain for $\theta \geq -\sigma/2$,

$$\max_x E[\Pi_T(x)] \approx \max_x \left\{ [R - \exp(u + x)\sigma]^+ \left[\Phi\left(\frac{x - \theta \mathrm{T}}{\sqrt{\mathrm{T}}}\right) + e^{2\theta x} \Phi\left(\frac{x + \theta \mathrm{T}}{\sqrt{\mathrm{T}}}\right) \right] \right\} \tag{15}$$

Let $x^* = \arg\max E[\Pi_T(x)]$. Then from section 2, we can derive that the optimal purchase price is $\exp[x^* + (r + h)t - hT]$.

3.2.3. Strategy Comparison under the Fully Dynamic Case

Proposition 3.2. *In fully dynamic cases, when $\theta \leq -\sigma/2$, the Target Strategy is no better than the Time Strategy at time T, which is purchasing at the end when the profit at that point is positive.*

Proof:

$$\begin{aligned}
E[\Pi_T(x)] &= \{E[R - \exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \leq T]^+ P(\tau_\theta(x) \leq T) \\
&\quad + E[R - \exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T]^+ P(\tau_\theta(x) > T)\} \\
&\leq [R - E(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \leq T)]^+ P(\tau_\theta(x) \leq T) \\
&\quad + [R - E(\exp(u + w_\theta(T)\sigma \mid \tau_\theta(x) > T)]^+ P(\tau_\theta(x) > T) \\
&\leq [R - E(\exp(u + w_\theta(T)\sigma \mid \tau_\theta(x) \leq T)]^+ P(\tau_\theta(x) \leq T) \\
&\quad + [R - E(\exp(u + w_\theta(T)\sigma \mid \tau_\theta(x) > T)]^+ P(\tau_\theta(x) > T) \\
&= [R - E(\exp(u + w_\theta(T)\sigma)]^+.
\end{aligned}$$

If $\theta \leq -\sigma/2$, $E[\Pi_T(x)] \leq E[\Pi(T)^+]$, so the better strategy is to purchase at $t = T$, i.e., when price is falling, it is better to wait until the end to purchase.

Proposition 3.3. *In fully dynamic cases, the expected profit of the Target Strategy is better than or equal to the expected profit of purchasing right away.*

Proof:

For the Target Strategy, if we set the purchase level $x = 0$, we will get the same result as purchasing at time 0.

Next we will find when the expected profit of the Target Strategy exceeds that of the Time Strategy. From (15),

$$\max_x E[\Pi_T(x)] = \max_x \left\{ (R - e^{(u+x)\sigma})^+ \left[\Phi\left(\frac{x - \theta T}{\sqrt{T}}\right) + e^{2\theta x} \Phi\left(\frac{x + \theta T}{\sqrt{T}}\right) \right] \right\}.$$

If the expected profit at time 0 satisfies $R - e^{u\sigma} < 0$, Proposition 3.3 is true because $\max_x E[\Pi_T(x)] \geq 0 > R - e^{u\sigma}$.

If $R - e^{(u+x)\sigma} \geq 0$ for $x \leq 0$,

let $G(x) = (R - e^{(u+x)\sigma}) \left[\Phi\left(\frac{x - \theta T}{\sqrt{T}}\right) + e^{2\theta x} \Phi\left(\frac{x + \theta T}{\sqrt{T}}\right) \right]$.

By taking the derivative of $G(x)$, we get

$$\begin{aligned} G(x)_x' = {} & (R - e^{(u+x)\sigma})\left[\frac{\phi((x - \theta T)/\sqrt{T})}{\sqrt{T}} + e^{2\theta x} \frac{\phi((x + \theta T)/\sqrt{T})}{\sqrt{T}} + 2\theta e^{2\theta x} \Phi\left(\frac{x + \theta T}{\sqrt{T}}\right)\right] \\ & - \sigma e^{(u+x)\sigma} \left[\Phi\left(\frac{x - \theta T}{\sqrt{T}}\right) + e^{2\theta x} \Phi\left(\frac{x + \theta T}{\sqrt{T}}\right)\right], \; where\ \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \end{aligned} \tag{16}$$

In (16), setting $x = 0$, we get

$$G(0)_x' = (R - e^{u\sigma})\left[\frac{\sqrt{2}\, e^{-\frac{t\theta^2}{2}}}{\sqrt{\pi}\, T} + 2\theta \Phi(\theta\sqrt{T})\right] - \sigma e^{u\sigma}. \tag{17}$$

When (17) < 0, we can get

$$\frac{1}{Te^{\frac{T\theta^2}{2}}} < \sqrt{\frac{\pi}{2}}[\frac{\sigma e^{u\sigma}}{R-e^{u\sigma}} - 2\theta\Phi(\theta\sqrt{T})]. \qquad (18)$$

If $\theta\Phi(\theta\sqrt{T}) > \frac{\sigma e^{u\sigma}}{2(R-e^{u\sigma})}$, the right side of (18) is negative, implying (17) > 0, which means the optimal solution for the Target Strategy is to purchase at $x = 0$, which is to purchase at time 0. From this condition, we notice that when θ is relatively large, or σ is relatively small, and the profit at time 0, which is $R - e^{u\sigma}$, is relatively big, it is better to purchase at the beginning. This is consistent with common sense.

If $\theta\Phi(\theta\sqrt{T}) < \frac{\sigma e^{u\sigma}}{2(R-e^{u\sigma})}$ and $\sqrt{\frac{2}{\pi}}te^{\frac{Tt\theta^2}{2}} > \frac{(R-e^{u\sigma})}{\sigma e^{u\sigma} - 2\theta\Phi(\theta\sqrt{T})(R-e^{u\sigma})}$, $G_0(x)' < 0$ is supported. In this case, we can get that (17) < 0. That implies that at point 0, the total profit follows an increasing trend; so that the optimal purchase level is lower than 0. So in this situation, the Target Strategy is better than the Time Strategy.

From the above analysis, we find that if there is an upper trend, we maximize our profit by taking the risk only when the current profit is relatively small or even negative, price fluctuation is relatively big, and the time to observe and make the decision is relatively long; otherwise, it is not worth taking the risk and we should purchase right away.

In the time interval $[0,T]$, if $t^* = \arg\max_{0\le t\le T} E\left[\Pi(t)^+\right] = 0$, i.e., the optimal "time" is to purchase at the beginning. From Proposition 3.3, we get that the optimal solution of the fully dynamic case is to purchase at specific level $x^* = \arg\max E\left[\Pi_T(x)\right]$ during $[0,T]$.

If $t^* = \arg\max_{0\le t\le T} E\left[\Pi(t)^+\right]$, and $0 < t^* \le T$, we can not tell which one is better, the Time Strategy or the Target Strategy, without the exact parameter values. So our strategy is to compare $\max E\left[\Pi(t)^+\right]$ and $\max E\left[\Pi_T(x)\right]$. If $\max E\left[\Pi(t)^+\right] \geq \max E\left[\Pi_T(x)\right]$, we will choose to wait until t^* and apply the Target Strategy after that, otherwise we will purchase at a specific level $x^* = \arg\max E\left[\Pi_T(x)\right]$ during $[0,T]$.

If $\frac{d[E\,\Pi(t)^+]}{d\,t} > 0$ for all $t \in (0,T]$, the optimal "time" is to purchase at the end. Specifically as $\theta < -\sigma/2$, from proportion 3.1, the optimal strategy is to purchase using "time" where $t = T$.

In the model, θ is used to include all the information about interest rate, price trend, and holding cost rate and σ is used to describe the fluctuation of total discounted cost trend. From the above analysis, we get that if θ and σ keep constant, as $\theta < -\sigma/2$, the optimal strategy is to purchase using the Time Strategy where $t^* = T$. Otherwise, we calculate the optimal solutions for the Target Strategy and the Time Strategy to see if it is the right time and right level to purchase right away.

3.2.4. Dynamic Target Strategy under the Fully Dynamic Case

Note that the Target Strategy and analysis previously described refer to the decision at time 0. On the other hand, if the firm has not purchased at time 0, the Dynamic Target Strategy allows it to incorporate new information.

The Dynamic Target Strategy sets a new target level x as time processes. When $\theta < -\sigma/2$, the Time Strategy dominates the Target Strategy, so we wait until the end of the time horizon to consider purchasing in that case. Otherwise, the initial target x is set at the point where the basic Target Strategy first provides a better profit than the Time Strategy.

We then update the target level if it has not been reached in the previous period. The Dynamic Target solution strategy is presented below.

<u>*Dynamic Target Strategy Algorithm*</u>

If $\theta \geq -\sigma/2$
Then *For* $i = 1,2,...T-1$
Do

Let $T_i = T - i$, $u(i) = \dfrac{\log(p(i)) + hT_i}{\sigma}$,

$x^* = \arg\max_x E[\Pi_T(x) \mid u(i), T_i]$, $y = e^{x^* + (r+h)t - hT_i}$

If $\max_x E[\Pi_T(x) \mid u(i), T_i] > \max_{i \leq t \leq T_i} E[\Pi(t)^+]$

and the price level y is reached in period i,

Then *Purchase at period i and stop.*

Else *Wait until the next period (set i=i+1)*

Loop

Else *Wait until the last period (set i=T)*

Purchase at i=T if $\Pi(T) > 0$.

In the appendix, we show that the profit obtained from the Dynamic Target Strategy is very close to optimal.

3.3. Timing Flexibility Case

In the timing flexibility case without quantity flexibility, the buyer must purchase the contracted quantity, without regard to expected profit. In the following propositions we illustrate that the Timing Strategy dominates the Target Strategy. Furthermore, the Dynamic Target Strategy reaches to the same result as the Time Strategy, *i.e.,* purchase at time 0 if $\theta > -\sigma/2$, purchase at any time between $[0,T]$ if $\theta = -\sigma/2$, and purchase at time T if $\theta < -\sigma/2$.

Proposition 3.4. *In timing flexibility cases, the expected profit of the Time Strategy at time* $t \in (0,T]$ *is* $E\,\Pi(t) = R - \exp(u\sigma + \theta\, t\sigma + \frac{t\sigma^2}{2})$.

Proof:

$$
\begin{aligned}
E\Pi(t) &= R - Ee^{[u+\theta t + w(t)]\sigma} \\
&= R - \int_{-\infty}^{+\infty} e^{(u+\theta t+\sqrt{t}x)\sigma} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\
&= R - e^{u\sigma + \theta t\sigma + t\sigma^2/2}
\end{aligned}
$$

Proposition 3.5. *In timing flexibility cases, the optimal decision by using the Time Strategy during* $[0,T]$ *is to purchase at time 0 if* $\theta > -\sigma/2$, *purchase at any time between* $[0,T]$ *if* $\theta = -\sigma/2$, *purchase at time* T *if* $\theta < -\sigma/2$.

Proof:

$$
\begin{aligned}
&\max_{t=[0,T]} E\Pi(t) \\
&= \max_{t=[0,T]} [R - e^{u\sigma+\theta t\sigma + t\sigma^2/2}]
\end{aligned}
$$

The optimal condition depends only on θ and σ, the Time Strategy for timing flexibility case is to purchase at time at time 0 if $\theta > -\sigma/2$, purchase at time T otherwise.

Proposition 3.6. *In timing flexibility cases, the optimal decision by using the Target Strategy is never better than the optimal decision by using the Time Strategy.*

Proof:

Let $\tau_\theta(x) = \inf\{t : t \geq 0; \theta t + w(t) = x\}$; $x \leq 0$, $W_\theta(t) = \theta\, t + w(t)$

We use $\Pi^t(x)$ to represent the Target Strategy with level x.

Then the optimal profit for the Target Strategy is:

$$
\begin{aligned}
\max_x \; E[\Pi^t(x)] \le \max_x \{ & [R - E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T\big)]P(\tau_\theta(x) \le T) \\
& + [R - E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) > T\big)]P(\tau_\theta(x) > T)\} \\
\le \max_x \{ & [R - E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T\big)]P(\tau_\theta(x) \le T)\} \\
& + [R - E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) > T\big)]P(\tau_\theta(x) > T)
\end{aligned}
$$

If $\tau_\theta(x)$ is inside $[0,T]$, when $\theta \le -\sigma/2$, from time $\tau_\theta(x)$ to T, referring to the result from part (1), the minimum $Ee^{[u+w_\theta(t)]\sigma}$ should be at time T, that is

$$
\begin{aligned}
& E\big(\exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) \le T\big) \ge E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T\big) \\
& \max_x \; \Pi^t(x) \le \max_x \{[R - E\big(\exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) \le T\big)]P(\tau_\theta(x) \le T) \\
& \qquad + [R - E\big(\exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T\big)]P(\tau_\theta(x) > T)\} \\
& \qquad = \max_x \{R - E\exp[u + w_\theta(T))\sigma]\} \\
& \qquad = \{R - E\exp[u + w_\theta(T)\sigma]\}
\end{aligned}
$$

If $\theta > -\sigma/2$, from time $\tau_\theta(x)$ to T, referring to the result from part (1), the minimum $Ee^{[u+W_\theta(t)]\sigma}$ should be at time $\tau_\theta(x)$, that is

$$
\begin{aligned}
& E\{\exp(u + w_\theta(T))\sigma \mid \tau_\theta(x) > T\} \ge E\{\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) > T\} \\
& \max_x \; \Pi^t(x) \le \max_x \{[R - E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) \le T\big)]P(\tau_\theta(x) \le T) \\
& \qquad + [R - E\big(\exp(u + w_\theta(\tau_\theta(x)))\sigma \mid \tau_\theta(x) > T\big)]P(\tau_\theta(x) > T)\} \\
& \qquad = \max_x \{R - E\exp[u + w_\theta(\tau_\theta(x)))\sigma]\} \\
& \qquad = \{R - E\exp[u + w_\theta(0)\sigma]\}
\end{aligned}
$$

Therefore, the optimal decision for a timing flexibility case using the Target Strategy is no better than the Time Strategy.

In section 2, we define $\theta = \dfrac{\mu - r - h - 0.5\sigma^2}{\sigma}$, so $\theta > -\sigma/2$, $\theta = -\sigma/2$ and $\theta < -\sigma/2$ can be written as $\mu - r - h > 0$, $\mu - r - h = 0$, and $\mu - r - h < 0$ correspondingly. Here

$\mu - r - h$ can be interpreted as the "cost slope", which is the slope of price trend minus the annual interest rate and the unit holding cost. So we can conclude that in the timing flexibility case where the contracted amount must be purchased, the firm should purchase right away if there is an upper cost trend, purchase at any time if the cost has no trend, and purchase at the end if there is a downward cost trend.

3.4. Combined Timing and Quantity Flexibility Contract

The strategy for the combined timing and quantity flexibility contract is composed of two parts: $(1-\alpha)Q$ purchased from the supplier using a timing flexibility contract, and αQ using a fully dynamic contract. The results can be summed up as follows.

Table 2. Strategies for the Combined Timing and Quantity Flexibility Contract

	If $\theta \le -\sigma/2$	If $\theta > -\sigma/2$
$(1-\alpha)Q$	Purchase at the beginning	Purchase at the end
αQ	Dynamic Target Strategy	Purchase at the end if the profit at the end is positive

3.5. Numerical Analysis

In this section we present numerical analysis and simulation results to test how the different strategies perform under various parameter settings. For all the tests we let T =100 and use a demand of 1000 units. Here we assume σ =0.1. At time 0 the selling price for these 1000 units is $e^{12.8} = \$362{,}217$ and the purchase cost for these 1000 units is $e^{12.4} = \$242{,}801$. The parameter value θ is given in Table 3. We run the simulation 100 times and calculate the averages as the simulation results.

In the simulation, the computational time of the Dynamic Target Strategy method averaged 3.2 seconds, whereas the backward deduction method using a binomial approximation averaged about 4,725.47 seconds (We divided each period into 10 time intervals to approximate the price process).

Table 3. Profit Information											
	Timing Flexibility Contract		**Fully Dynamic Contract**								
	Time Strategy		**Time Strategy**		**Target Strategy**		**Dynamic Target Strategy**		**Backward Deduction**	**Upper Bound**	
θ	t^*	$\hat{\Pi}(t^*)$	t^*	$\hat{\Pi}(t^*)^+$	Prob. of purchasing	$\hat{\Pi}_T(x^*)$	Prob. of purchasing	$\hat{\Pi}$			
.03	0	119,416	0	119,416	.81	123,424	.91	127,256	119,416	166,322	
.02	0	119,416	0	119,416	.82	137,736	.85	140,054	130,010	167,735	
.01	0	119,416	0	119,416	.75	134,043	.87	144,044	132,121	175,576	
.00	0	119,416	0	119,416	.72	145,734	.81	148,763	139,274	179,060	
-.02	0	119,416	50	138,113	.76	148,824	.84	150,142	144,139	182,556	
-.05	0	119,416	50	185,766	.76	168,227	.86	185,766	185,766	194,528	
-.08	50	181,775	50	232,299	.80	219,267	.90	232,299	232,299	247,407	

Table 3 provides simulation results for the optimal time and the corresponding profit for the Time Strategy under both a timing flexibility contract and a fully dynamic contract, the probability of purchasing and the corresponding profit for the Dynamic Target Strategy, the profit using the backward deduction method, and the upper bound which is the average lowest profit of the history data of these 1000 price lines. From the table we notice that the Dynamic Target Strategy always performed at least as well as the backward deduction method. (Note that the profit of the backward deduction profit does increases in the number of time intervals used per period).

For fully dynamic contracts, when $\theta \geq -\sigma/2$, the Target Strategy outperformed the Time Strategy while the Dynamic Target Strategy was the best. When $\theta < -\sigma/2$, purchasing at time T , which provides the same result as the Dynamic Target strategy, was always the best solution. Therefore the results were consistent with our analysis.

4. Extensions to the Multiple Suppliers

When the price processes differ between suppliers, the retailer can benefit by comparing their price movements and selecting the lowest cost supplier. The price processes for multiple suppliers may differ for a variety of reasons. For example, their inputs might derive from different commodities (e.g., aluminum vs. steel), or they might come from different countries where the exchange rate fluctuations impact their corresponding price movements differently.

In this section, we examine the general case of choosing among multiple suppliers under three objectives: (1) maximizing expected profit, (2) minimizing downside risk, and (3) maximizing profit subject to a limit on downside risk. We assume that each potential supplier has enough capacity to satisfy the retailer's demand and that each offers a fully dynamic contract.

4.1. Profit Maximization

Given n suppliers, the corresponding u , θ and σ for supplier i are u_i , θ_i and σ_i for $i = 1,2...,n$. We first study how to extend the Target Strategy to the multiple suppliers case. We search for a common targeted price for all the suppliers so that the expected profit, which is the product of the profit at this cost and the probability of any of these suppliers' prices reach this target price, is maximized. If any supplier's unit profit reaches the target profit during $[0,T]$, we will purchase D from this supplier right away. The corresponding expected profit can then be described as:

$$E\Pi_M(x) = (R - \exp x)\left[1 - \Pi_{i=1}^{n} P\left(M_{\theta_i}(t) > \frac{x}{\sigma_i} - u_i\right)\right]. \qquad (19)$$

Because $1 - \Pi_{i=1}^{n} P\left(M_{\theta_i}(t) > \frac{x}{\sigma_i} - u_i\right) \geq \max_{i, 1\leq i\leq n} P\left(M_{\theta_i}(t) < \frac{x}{\sigma_i} - u_i\right)$, the Target Strategy for multiple suppliers provides a better expected profit than solely applying the Target Strategy to all of the suppliers individually. Thus, the expected profit increases in the number of suppliers *n*.

Next, we can extend the Dynamic Target Strategy to the multiple suppliers case. We set a new target level x for all suppliers as time progresses. The initial target level x is set at the point where the Target Strategy using (19) can first provide a better profit than the Time Strategies of any supplier. We then update the target level if it has not been reached by purchasing from any supplier in the previous period. Note when $\theta_k < -\sigma_k/2$, $\forall k \in [1,n]$, even if purchasing from supplier i reaches the target profit, from proposition 3.2, we see that a higher expected profit is achieved by waiting until the end of the period.

The Dynamic Target solution strategy for n suppliers is presented below. Here use $p_j(i)$, $u_j(i)$, and $\Pi^{\sup j}(t)$ to represent the price, u-value, and the unit profit of supplier j at time t correspondingly.

<u>*Dynamic Target Strategy Algorithm for n Suppliers*</u>

For $i = 1,2,...T-1$
Do

Let $T_i = T - i$, $u_j(i) = \dfrac{\log(p_j(i)) + hT_i}{\sigma}$ *for* $j = 1,2...n$;

$$x^* = E\Pi_M^{-1}\left\{\max\left[\max_x E[\Pi_M(x)], \max_{1\le j\le n}\max_{i\le t\le T} E[\Pi^{\sup j}(t)^+]\right]\middle| T_i, u_j(i), j=1,2...n\right\},$$

$\mathrm{y}_{\mathrm{j}} = e^{x^*+(r+h_j)t-h_jT}$ *for* $j=1,2...n$.

If y_k *is reached by supplier* k

and $E[\Pi_M(x^*)] > \max_{i\le t\le T} E[\Pi^{\sup k}(t)^+]$

Then Purchase from supplier k *at period* i *and stop.*

Else Wait until the next period (set i=i+1)

Loop

Purchase from supplier k *at i=T if* $\Pi^{\sup k}(T)^+ = \max\{\Pi^{\sup j}(T)^+, \forall j \in [1,n], 0\}$.

Here $R - \exp x$ is the target unit profit for the retailer, $\left[1-\Pi_{i=1}^n P\left(w_{\theta_i}(t) > \frac{x}{\sigma_i} - u_i\right)\right]$ is the probability of reaching this target unit profit and can be calculated using (9).

4.2. The Probability of Reaching Target Profit

We consider the downside risk using a certain supplier as the probability that the realized profit is less than or equal to the retailer's specified target unit profit (Gan, Sethi, and Yan, 2005). Let γ be the target unit profit. Clearly, the optimal strategy to minimize the downside risk is to purchase when the profit is bigger than γ during $[0,T]$. From the definition of the Target Strategy, we can obtain that the downside risk of choosing supplier 1 to n at target unit profit γ is

$$p[\Pi_M(x) \le \gamma] = \Pi_{i=1}^n P\left(w_{\theta_i}(t) > \frac{\log(R-\gamma)}{\sigma_i} - u_i\right) \qquad (20)$$

To minimize the risk expressed in (20), the corresponding optimal purchasing strategy is to purchase right away from supplier i when $\Pi_M(x) > \gamma$, *i.e.*, when the price of supplier i first drops below $y = (R-\gamma)\exp[(r_i + h_i)t - h_iT]$, i=1, 2 … or n.

4.3. Maximizing Profit Subject to a Downside Risk Limit

With a downside risk limit τ, the company's decision problem becomes:

$$\max E(\Pi)$$

$$\text{s.t. } p\big(E(\Pi) \le \gamma\big) \le \tau$$

According to Gan, Sethi, and Yan (2005), the target level γ could be associated with bankruptcy or something less drastic.

If we use the Dynamic Target Strategy to solve this problem, then the above problem can be written as:

$$\max E\Pi_M(x) \tag{21}$$
$$\text{s.t. } p\big[E\Pi_M(x) \le \gamma\big] \le \tau \tag{22}$$

The approximate solution is to purchase D from the supplier whose price first drops below price c (defined below) during $[0, T]$. If no purchase has been maded before time T and have not yet purchased, the buyer should purchase D from supplier i, where $\Pi^{\sup i}(T)^+ = \max\{\Pi^{\sup j}(T)^+, \forall j \in [1, n], 0\}$.

Let $x^* = E\Pi_M^{-1}\left\{\max_{i \le t \le T} E[\Pi^{\sup j}(t)^+],\ j = 1,2\ldots n;\ \max_x E[\Pi_m(x)] \mid T_i, u_j(i)\right\}$, then c is decided by:

$$c = \begin{cases} e^{x^* + (r+h)t - hT} & \text{if } p\big[E\Pi_M(x) \le E\Pi_M(x^*)\big] \le \tau \;\&\; E\Pi_M(x^*) \ge \gamma \\ e^{y^* + (r+h)t - hT} & \text{if } y^* = \arg\max_y \big\{p\big[E\Pi_M(x) \le y\big] \le \tau\big\} \;\&\; \gamma \le y^* < E\Pi_M(x^*) \\ \text{no solution} & \text{Otherwise} \end{cases}$$

$E\Pi_M(x)$ and $p\big[E\Pi_M(x^*) \le y\big]$ can be calculated referring to (19) and (20).

The intuition for this solution is the following. If the optimal profit by using (21) is higher than γ and $p\big[w_\theta(T) < x^*\big] \le \tau$, (22) is satisfied by the optimal solution from (21). Otherwise, we will try to find the lowest target price so that the probability of not attaining the corresponding profit is no more than τ and the corresponding profit is the highest profit we can get. If (22) is not satisfied, the downside risk constraint cannot be reached. Note this is only a roughly estimation and the real profit should be higher than this result because we use the first target profit to estimate the profit here, where the profit using the Dynamic Target Strategy should be higher than this.

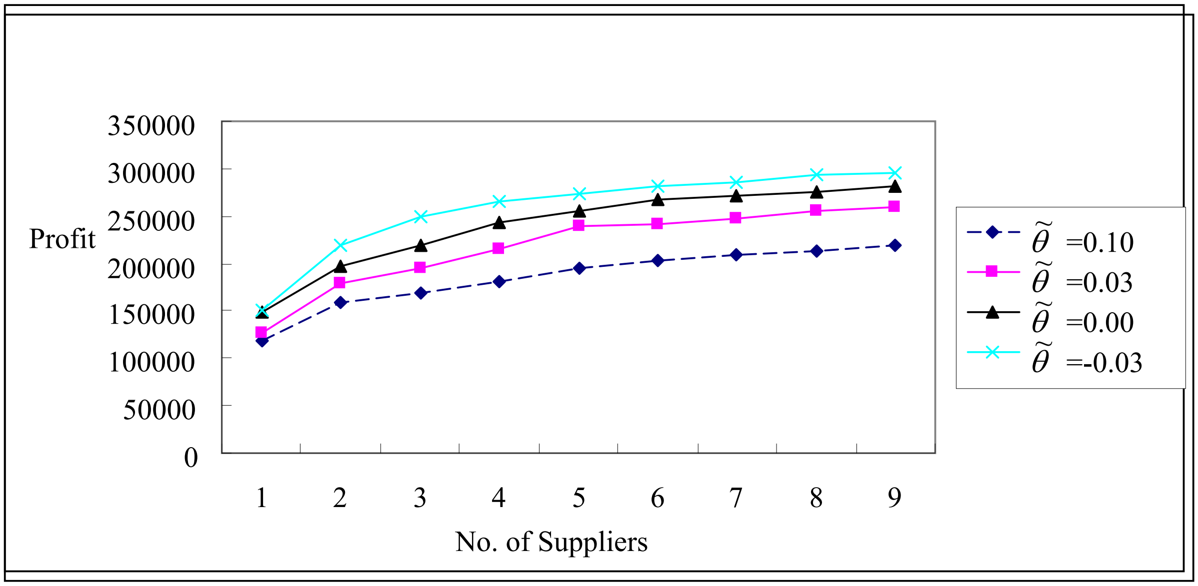
4.4. Numerical Analysis with Multiple Suppliers

4.4.1. Profit Impact of Adding Potential Suppliers

Use the same assumption of 3.5, we let T=100 and use a demand of 1000 units. But we further assume there are n suppliers, $n = 1,2,3...,9$. For all n suppliers, $p_1(0) = p_2(0) = ... = p_n(0)$, $\sigma_1 = \sigma_2 = ... = \sigma_n = 0.1$, and $\theta_1(0) = \theta_2(0) = ... = \theta_n(0) = \tilde{\theta}$, where $\tilde{\theta}$ is as defined in the Plot 4.1. At time 0 the selling price for these 1000 units are all $e^{12.8} = \$362,217$ and the purchase cost for these 1000 units is $e^{12.4} = \$242,801$. We run the simulation 100 times and calculate the averages as the simulation results. Figure 7 shows how the profit increases using the above method when the number of suppliers increases from 1 to 9.

From this plot, we can see as the number of suppliers increases, the profit increases, and the increasing of profit is steeper when the number of suppliers is smaller. Furthermore, when the number of suppliers increases from 1 to 9, the profits are all almost doubled.

Figure 7. Profit Trend vs the No. of Suppliers

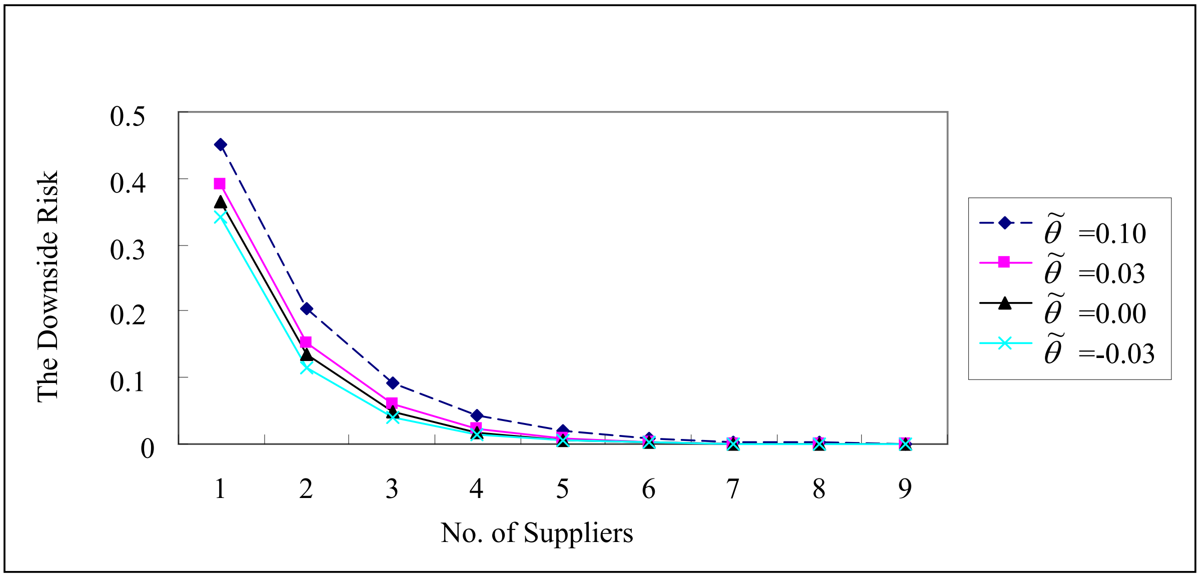


4.4.2. Risk Impact of Adding Potential Suppliers

Using the example in 4.4.1, we assume the target profit for these 1000 units is $150,000. Then the optimal purchasing strategy is to purchase right away when the price of any supplier drops below. The following plot shows how the corresponding downside risks drops as the number of suppliers increases.

From the plot, we can see that decrease in downside risk is very sharp when the number of suppliers is small while the decrease in downside risk is much flatter when the number of suppliers is bigger. No matter what the cost trend θ is, for this example, choosing 4 suppliers can decrease the downside risk to lower than 5%, so choosing 4 suppliers will be enough if we want to control our downside risk to 5% in this case.

Figure 8. The Downside Risks vs the No. of Suppliers



4.4.3. An Example Applying the Three Optimization Criteria

Let us look at one round of one specific case of the above example with θ=0.1 and four suppliers. With the purpose of profit maximization, the buyer reaches his satisfied profit at time 5, and his profit is \$215,162. With the purpose of profit maximization, the buyer reaches his satisfied profit at time 3, and his profit is 147,945. With the purpose of profit maximization subject to a downside risk limit $p[\Pi_M(x) \le 150)] < 5\%$, the buyer reaches his satisfied profit at time 4, and his profit is 169,609. How much the profit with the purpose of profit maximization subject to a downside risk limit is depends on how strict the downside risk limit is.

5. Conclusions

In this chapter, we study combined timing and quantity flexibility contracts under price uncertainty. Even though a binomial tree approximation is used extensively to solve price uncertainty problems, the solution time of that method increases exponentially with the number of the suppliers and the number of periods. Therefore we propose another solution algorithm for this specific problem that substantially decreases the computational complexity.

As a byproduct of the analysis, we demonstrated that in the case of time flexibility without quantity flexibility, the best strategy is to buy at the beginning or the end depending on the total cost trend. In this case, the price fluctuation σ does not influence the purchasing strategy. On the other hand, for fully dynamic contracts, our Dynamic Target Strategy appears to perform best. That strategy utilizes a profit target, which is updated over time as conditions change.

When facing multiple potential suppliers with potentially different price processes, the analysis of timing flexible cases does not differ much from the single-supplier case. But if there is quantity flexibility available, purchasing from multiple suppliers with potentially different price processes can lead to higher profits due to a higher probability of being able to buy at a low price from at least one of the suppliers. We develop a solution algorithm for the case of multiple suppliers. We further provide downside risk analysis and study how to maximize profit subject to a downside risk limit for the multiple supplier case to provide multiple criteria for the selection of suppliers. We demonstrate the profound impact that increasing the potential supplier base can have on profit and downside risk.

CHAPTER IV: THE NEWSVENDOR PROBLEM UNDER PRICE-SENSITIVE STOCHASTIC DEMAND AND PRESEASON PURCHASE PRICE UNCERTAINTY

1. Introduction

As a consequence of price fluctuations in spot markets, a pure fixed-price contract also rarely exists. If the actual work or spot price varies from estimates, most of the cases, the client will pay the difference (Metagroup 2003) or the supplier and retailer share the risks (Li and Kouvelis 1999). This is especially true for high-volume commodities such as gasoline and natural gas (Avery et al.1992). So no matter the company makes a purchase from the spot market or the contracted suppliers, purchase-price risk commonly exists. Systematic purchase-price risk is the major financial risk to consider in inventory control (Berling and Rosling 2005).

On the one hand, as purchase costs of the components fluctuate, the company needs to adjust the selling price of the product to optimize profit under new costs. On the other hand, as a result of the changes in selling price, the customer demand will be negatively affected. For example, the fuel cost increases led to increased air fares. But a significant portion of the fuel cost increase is being borne by the customers, because for many passengers, particularly leisure travelers, significant fare increases make the difference between taking a trip and staying at home. In order to handle the increase in fuel cost, a lot of the airlines like American Airlines, Delta, and Southwest, have already purchased futures for those fuels to hedge their prices as time goes on. For example, 55% of the fuel used in 2000 by Delta was purchased in 1997 (Schriver, 2008).

Facing high component price uncertainty, forward purchasing is commonly used to hedge procurement risks; however, forward purchasing itself may lead to significant "incremental" risk. For example, in mid 2000, HP signed a long-term binding contract with a major supplier to actively manage the substantial future price uncertainty of Flash Memory. It turned out that HP was put in a disadvantageous position while Flash Memory prices dropped; HP paid more through that fixed- price commitment than its competitors. To ensure that such increased risks stemming from forward contracts were minimal and more manageable in the future, it was necessary to compare, in detail, the demand and price uncertainty scenarios for Flash memory with the quantity and price HP committed to in the contract. In the 2002 report of the Procurement Risk Management Group at HP, the researchers concluded that "the long-term binding contract for Flash memory signed in mid-2000 set the course for the active management of procurement uncertainties and risks in HP".

The above business practices bring to light three important research questions which we plan to address in this chapter. First, we study how to plan the procurement and selling for a newsvendor who has a specific time period prior to selling season to make the purchase, where it is assumed that the purchasing cost of the raw material fluctuates over time and the demand for the product is random and price-sensitive. Second, we analyze how to evaluate and compare the potential profit and risk of purchasing from a forward contract as opposed to purchasing from the spot market. Third, we explore how to evaluate and minimize risk in the spot market.

This chapter makes three primary contributions. First, it is among the first to combine purchase-order-quantity, purchase-order-time, and selling-price decisions all together where purchase prices fluctuate over time and demand is random and price-dependent. Second, we design practical and efficient solution procedures for purchasing solely from the spot market, and for choosing between the spot market and a forward contract, which quickly decreases the computational complexity in solving the computational problem of multiple suppliers, long term length, and multiple demand points. Third, we apply risk analysis to the procurement and selling problems under settings of stochastic customer demand and purchase price uncertainty.

The rest of the chapter is organized as follows: In the following section, we present the relevant literature. Section 3 addresses the basic profit model for optimal one-time purchasing strategies in the Newsvendor problem with purchasing before the selling season; this section also finds the optimal pricing policy in closed form, and provides an upper bound on the expected cost before the selling season. We extend our analysis to study structural properties for purchasing from spot markets and estimate the profit and risk when multiple suppliers or multiple periods are available in section 4. In section 5, we provide numerical analysis to decide when to purchase and how much to order if we plan a second purchase option during the horizon, compare the profit and strategy of purchasing at the lowest expected cost with the profit and strategy of purchasing at the highest expected profit, evaluate the impact of the unit holding cost rate on the purchasing decision, and demonstrate the profound impact that increasing the potential supplier base can have on profit and risk, together with the effect of these parameters on the purchasing decision. We conclude in the last section with some final observations.

2. Literature Review

One relevant research stream analyzes how stochastic purchase price affects inventory policy. Bjerksund and Steinar (1990) discuss project values and operational decision rules by interpreting investment as an option and the output price as the underlying asset. Their paper compares the present price with the expected price at any future time to decide on purchase right away or choose to wait and see. Li and Kouvelis (1999) develop optimal purchasing strategies for both time-flexible and time-inflexible contracts with risk-sharing features in environments of price uncertainty. They also expand the analysis to two-supplier sourcing environments and quantity flexibility in such contracts. Berling and Rosling (2005) study how to adjust (R, Q) inventory policies under stochastic demand or stochastic purchase costs. In those models, the inventory decision under stochastic purchase costs is studied under the assumption that demand and total order quantity are constant and that the purpose of their strategy is to lower the total cost without considering the sale. In this chapter, the demand is assumed to be stochastic; we combine the purchasing and selling into one whole and take into consideration not only the timing of the purchase but also the order quantity and the selling price to maximize the overall profit.

Another kind of relevant literature is newsvendor-pricing problems with random price-dependent demand function. Most published papers study single period problems. Among those papers, Mills (1959) studies additive demand cases like we do in this chapter. Others like Karlin and Carr (1962) study multiplicative case, and Young (1978) combined additive and multiplicative effects. Petruzzi and Dada (1999) provide a comprehensive review of single period price-dependent demand newsvendor literatures, and further develop additional results to enrich the existing knowledge base. Theirs is the paper most related to our models. Petruzzi and Dada (1999), Chan, Simchi-Levi, and Swann (2001), and Monahan and Petruzzi (2004) extend the analysis of newsvendor-pricing problems to multiple period cases. However, in these papers, the purchase cost is assumed to be fixed and the main concern is how to design selling strategies without considering the purchasing side.

A third stream of relevant literature, studies risk-control and the inventory decisions of risk-averse firms. In the field of economics and finance, agents are often assumed to be risk-averse (Agrawal and Seshadri 2000). There are also some attempts at dealing with risk control and risk aversion in the operations area. For example, Bouakiz, and Sobel (1992) and Eeckhoudt et al. (1995) study the inventory problem of a risk-averse firm. Buzacott et al. (2001) study how to make the price and inventory decision jointly for a risk-averse company. Among those papers, two common methods to deal with risk aversion are mean-variance analysis and downside risk analysis. Markowitz (1959) is the first person to induce the mean-variance method to evaluate risk aversion in investment. Others apply this method to inventory decisions or supply contracts design (Chen and Federgruen 2000, Martlnez-de-Albeniz and Simchi-Levi 2006). The mean-variance approach works best when the retailer's profit follows normal distribution. But this assumption does not fit the newsvendor setting. Therefore, the downside risk is more important than simply the variance of the profit for the newsvendor (Gan et al. 2005). Telser (1955) and Gan et al. (2005) define downside risk as a critical value for profit, a measure we borrow in this dissertation. There are a number of other measurements of downside risk, for example: semi-variance computed from mean return (SVM), semi-variance computed from a target return (SVT), and value at risk computed from a specific fractile of the return distribution

(VaR). Sample works about downside risk in the newsvendor setting include Telser (1955), Buzacott et al. (2001), Gan et al. (2005), and so on. These papers are concerned on how to prevent the profit from being lower than the expected return instead of how to prevent the profit from deviating from the expected return, so that downside risks fit these settings better.

In this chapter, demand uncertainty and purchase price uncertainty are combined together into one consideration and procurement and selling are considered together as an entire entity instead of independent problems. We further extend the previous research by first applying risk analysis to the procurement problem under stochastic demand and purchase price uncertainty settings. We want to find effective and efficient ways to simplify these highly complex problems companies face.

3. The Model

Consider the situation where a firm stocks a certain amount of a single product over a time period $[0,T]$ from one supplier and process it into the final product, facing a random price-dependent demand for the final product at selling season T .

This process is composed of two stages. The first stage is to purchase a product from the stochastic spot market or a contracted supplier whose price was impacted by the exchange rate movement and turn it into the final product during a time period $[0,T)$ and the second stage is the retailer selling the final product to the customer at time T. The corresponding total cost includes the fixed cost and the variable cost. We use k to represent the total fixed cost for both stages. Next, we will study the variable cost in both stages.

The variable cost in the first stage includes the unit purchasing cost, processing cost, and holding cost from the time of purchasing to the time of processing. Because these costs happen before selling season T, we name this total cost preseason cost.

To model the unit purchasing cost, we assume that the retailer will pay the supplier $a(t) > 0$ dollars per unit when the unit is purchased at time $t \in [0, T]$. As in chapter III, the purchase price per unit for the retailer satisfies the usual Black-Scholes equation (Black and Scholes 1973), *i.e.*, the price at time t is a process $a = \{a(t), t \geq 0\}$, which is expressed by the stochastic differential equation

$$\mathrm{d}a(t) = a(t)\{\mu \,\mathrm{d}t + \sigma \,\mathrm{d}W(t)\} \qquad t \geq 0, \tag{1}$$

where $\mu \in \mathbf{R}$ denotes the average rate and $\sigma \in \mathbf{R}^+$ represents the volatility rate.

As in chapter III, the unit purchase price process $a(t)$ at time t satisfies

$$a(t) = a(0)\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t) \right), \mu \in \mathbf{R},\ \sigma > 0 \text{ and } t > 0. \tag{2}$$

An important part of the preseason cost is the processing cost per unit. It may turn out to be the processing cost of turning the raw materials into final product for the production company or the transportation fee and packing fee for the retailer company. In this chapter, we assume that operation cost per unit and operation time are constant. Without loss of generality, we can take the operation cost as a constant part of purchasing cost $a(t)$ and will not study it separately; we also let the operation time to be zero.

If the firm purchases one unit at time t and uses it to satisfy the demand at time T, the holding cost for this unit is $a(t)[h(T-t)]$, where h is the periodic holding cost percentage. The preseason cost $c = \{c(t),\ t \geq 0\}$ per unit is expressed by

$$c(t) = a(t) + a(t)h(T - t) = a(t)(hT + 1 - ht) \tag{3}$$

The second stage is the retailer selling the final product to the customer at time T. Here we assume that the retailer charges p dollars per unit for the customer, and the demand is random and negatively influenced by the selling price p in the additive form $D(p,\varepsilon) = y(p) + \varepsilon$, where $y(p)$ is a downward sloping, concave, deterministic function of the unit selling price, and ε has a linear or log-concave density function $f(x)$ with mean μ and variance σ^2 and $x > A$. This additive demand form and the assumptions about $y(p)$ are common in the literature (Petruzzi and Data 1999).

To jointly decide purchasing time, order quantity, and selling price to maximize the total profit, we first look at the second stage and model as to how the optimal order quantity and selling price will depend on the preseason purchasing unit cost c. Next, we can look backwards at the first stage to find the optimal purchasing strategy and then the corresponding optimal order quantity and selling price.

3.1. The Selling Season Decisions

During the selling season, if the demand is less then the order quantity q, the leftovers $q - D(p,\varepsilon)$ are disposed of at the unit cost v and the revenue is $pD(p,\varepsilon)$. Alternatively, if demand exceeds q, then the shortage $D(p,\varepsilon) - q$ is penalized at the unit cost s and the revenue becomes pq . Here we use c to represent the realized preseason unit cost, then the total cost is the sum of the preseason cost and disposal or shortage cost, and the profit $\Pi(q, p \mid c)$ for a certain preseason cost c is the difference between sales revenue and the total costs. By substituting $D(p,\varepsilon) = y(p) + \varepsilon$ and consistent with Ernst (1970), defining $z = q - y(p)$:

$$\Pi(z, p \mid c) = p[y(p) + \varepsilon] - c[y(p) + z] - k - v[z - \varepsilon]^{+} I(\varepsilon \le z)$$

$$+ p[y(p) + z] - c[y(p) + z] - s[\varepsilon - z]^{+} I(\varepsilon > z) \qquad (4)$$

Note in the above equation, q and p are decision variables, while c relies on the evolution of purchase price process and the corresponding purchasing strategy. For each realized purchasing cost c , the corresponding optimal purchasing, stocking, and pricing policy is to stock $q^{*} = y(p^{*}) + z^{*}$ units with unit selling price p^{*} , where p^{*} and z^{*} maximize $E[\Pi(z, p \mid c)]$.

Expected profit is:

$$E[\Pi(z, p \mid c)] = \int_{A}^{z} \left(p[y(p) + x] - v[z - x]\right) f(x)dx$$

$$+ \int_{z}^{+\infty} \left(p[y(p) + z] - s(x - z)\right) f(x)dx - c[y(p) + z] - k \, ,$$

(5)

This equation can be rewritten as

$$E[\Pi(z, p \mid c)] = \varphi(p \mid c) - \psi(z, p \mid c) \, . \qquad (6)$$

Where

$$\varphi(p \mid c) = (p - c)[y(p) + \mu], \tag{7}$$

and

$$\psi(z, p \mid c) = (c + v)\int_A^z (z - x) f(x) dx + (p + s - c)\int_z^{+\infty} (x - z) f(x) dx + k\ . \tag{8}$$

Equation (6) shows that expected profit is expressed by the "riskless" profit, which would occur in the absence of demand uncertainty, less the expected loss that occurs as a result of demand uncertainty. Equation (7) is the "riskless" profit function (Mills 1959), in which ε is replaced by μ .Equation (8) is the loss function, which assesses overage cost and underage cost like the newsvendor problem with demand satisfying density function $f(x)$.

To find the maximized expected profit for a certain c, we take the first partial derivatives of $E[\Pi(z, p \mid c)]$ with respect to z and then p,

$$\frac{\partial E[\Pi(z, p \mid c)]}{\partial z} = p + s - c - (p + s + v)F(z)\ . \tag{9}$$

Thereby we solve the optimal value of z as a function of p for a given c :

$$z(p \mid c) = F^{-1}\left(\frac{p + s - c}{p + s + v}\right). \tag{10}$$

By substituting the result back into $E[\Pi(z, p \mid c)]$ and then searching over the resulting track to maximize $E[\Pi(z, p \mid c)]$, we get

$$p^*(c) = \left\{ p \mid \frac{\partial E\left[\Pi\left(p, z(p \mid c)\right)\right]}{\partial p} = 0 \right\}$$

$$\frac{\partial E[\Pi(p,z(p\,|\,c))]}{\partial p} = y(p) + \mu + (p-c)y^{'}(p) +$$

$$\frac{1}{f\left(F^{-1}\left(\dfrac{p+s-c)}{p+s+v}\right)\right)}\frac{c+v}{(p+s+v)^2}\left[(p+s-c)\int_{F^{-1}(\frac{p+s+c}{p+s+v})}^{+\infty} f(x)dx - (c+v)\int_{A}^{F^{-1}(\frac{p+s-c}{p+s+v})} f(x)dx\right] = 0 \tag{11}$$

Referring to the results in Whitin (1955) and Petruzzi and Dada (1999), the corresponding $z^*(c)$ and $p^*(c) = p(z^*(c)\,|\,c)$ by solving (10) and (11) are unique. However, a closed form expression is not obtainable in general (Porteus 1990) and we need to solve (11) using numerical analysis. Here we define $\Pi^*(c) = E[\Pi(z^*(c), p^*(c), c)]$. Then $\Pi^*(c)$ is a function uniquely decided by the preseason unit cost c. On the other hand, as the preseason purchase price fluctuates over time, the preseason unit cost c relies on both the purchase price and purchase time. So we need to catch the best purchasing chance to optimize the expected profit. Next, we analyze the evolution of the purchase price and then set up models to analyze how to decide on purchasing.

3.2. The Backward Solution Process

The optimal order quantity, as well as selling price, all depends on the preseason cost. However, because an important part of the preseason cost— the purchasing cost is an external factor, we cannot decide what the cost is but can only try to design optimal strategies to "catch" a purchase price so that this maximizes our total profit.

First, we design a backward deduction — a program to "catch" a purchase price and order quantity to maximize the expected profit over the whole horizon $[0,T]$ for the newsvendor with preseason procurement problem. To create this program, we introduce a binomial tree approximation method to buildup a discrete-time framework to approximate the evolution of $a(t)$. This method is suggested by Cox et al. (1979) and is a widely used numerical procedure to approximate the movement of fluctuating market price. The method is applied as follows: First, the time interval $[0, T]$ is divided into n small time intervals. $\Delta = T/n$, the unit purchasing cost $a(t+1)$, which represent the unit purchasing cost of the i^{th} node of the $(t+1)\Delta$ time interval of the binomial tree satisfies:

$$a_i[(t+1)\Delta] = \begin{cases} a_i(t\Delta)e^{\sigma\sqrt{\Delta}} & prob \quad \varsigma = \frac{1}{2}\left(1 + \frac{\mu - .5\sigma^2}{\sigma}\sqrt{\Delta}\right) \\ a_{i+1}(t\Delta)e^{-\sigma\sqrt{\Delta}} & prob \quad 1 - \varsigma = \frac{1}{2}\left(1 - \frac{\mu - .5\sigma^2}{\sigma}\sqrt{\Delta}\right). \end{cases} \tag{12}$$

After fixing the purchase price, we can get the preseason cost c for each node in each time interval of the binomial tree using (11).Then the corresponding optimal p and q for each c is calculated by (6) and (7), and the corresponding equation recursively backwards is:

$$U_i(t\Delta) = \max\left\{E\left(\Pi^*[a(t\Delta)(hT + 1 - ht\Delta)]\right), E_{i,t|t+1}\left(U[(t+1)\Delta]\right)\right\}, \tag{13}$$

$$U_i(t\Delta) = \max\left\{E\left(\Pi^*[a_i(t\Delta)(hT + 1 - ht\Delta)]\right), \varsigma U_i[(t+1)\Delta] + (1-\varsigma)U_{i+1}(t+1)\Delta]\right\} \tag{14}$$

Then the decision function is as follows:

If $\Pi^*[a_i(t\Delta)(hT + 1 - ht\Delta)] \geq \varsigma U_i[(t+1)\Delta] + (1-\varsigma)U_{i+1}(t+1)\Delta]$, the optimal decision at this node is to purchase with order quantity q^* immediately, the corresponding order quantity q^* and the potential selling price p^* are calculated using (6) and (7). If not, then it would be logical to wait until the next period and continue in this manner until the units have been purchased or the end of the time horizon has been reached.

The advantage of this method is that we can make the solution close enough to the optimal solution by dividing each time period into small enough intervals. For small preseason time length (<40) and single supplier, this method can be used to provide reliable result.

3.3. Strategy for Multiple Suppliers or Long Term Length Scenarios

In the above analysis, we employ the Binomial Lattice method to approximate the process of $a(t)$ in the way suggested by Cox et al. (1979) and then design a backward deduction heuristic to solve the purchasing and selling problems. However, reliable calculations for this heuristic method require each time period to be segmented into many small intervals, which significantly increases the computational burden for a large time horizon scenario. Furthermore, optimal selling price p^* and order quantity q^* are not obtainable in general (Porteus 1990) and we need to solve (11) using numerical analysis, which also adds to the calculation burden. What is more, when there are multiple suppliers, the number of calculations of all the combinations will evolve exponentially and closed form optimal solutions using the Binomial Lattice method will be almost impossible to compute. For example, if $T = 20$ and there are 3 suppliers, even assuming that a closed form expression for p^* and q^* is obtainable in general, the computational complexity is already about $(10^6)^3 = 10^{18}$. In this part, we study the properties of the model and of Brownian motion to develop a close-form solution procedure that significantly decreases the computational complexity, especially for the multiple-supplier case and large time horizon scenarios. Using the same-sized example described above, the computational complexity of our procedure is approximately 18,000.

To introduce this strategy, we assumed that the company chooses to purchasing in a later time instead of purchase right away only when the potential increase in profit overweighs the potential decrease in probability of reaching such a profit, *i.e.*, there exists a certain expected profit $E(\Pi^*)$ such that the product of $E(\Pi^*)$ and the probability of reaching $E(\Pi^*)$ by purchasing at a future procurement price is higher than the expected profit of purchasing at present procurement price. As in the backward deduction method, we first divide the time interval $[0,T]$ into n small observation time periods. $\Delta = T/n$, the unit purchasing cost $a(t+1)$ represents the unit purchasing cost of the i^{th} node of the $(t+1)\Delta$ time periods. At each new period, as long as the units have not yet been purchased, we recalculate the expected profit for purchasing at a certain price in each of the remaining periods in the time horizon. We compare the corresponding highest expected profit with the present profit. We will purchase right away if the highest expected profit drops below the present profit. If not, then we wait for the next period and continue in this manner until the units have been purchased or the end of the time horizon has been reached. Next, we introduce the detailed formula and process to calculate the results.

First, (10) can be approximated by $c(t) = a(t)(hT+1-ht) \approx a(t)(hT+1)\left[\exp\left(\frac{-ht}{hT+1}\right)\right]$.

Higher-order terms of this Taylor approximation quickly move towards zero when the holding cost is much smaller than the purchase price. Then

$$c(t) = a(0)(hT+1)\exp\left[\left(\mu - \frac{\sigma^2}{2} - \frac{h}{hT+1}\right)t + \sigma W(t)\right], \mu, r \in \mathbf{R},\ \sigma > 0 \text{ and } t > 0 \quad (15)$$

Subsequently, we shall focus on analyzing the $\log(c(t))$ instead of the actual discounted total cost process. Since $\log(c(t))$ is continuous and twice differentiable on $[0,T]\times\mathbf{R}$, it implies that $\log(c(t))$ is again an Ito process, and

$$\log c(t) = \log a(0) + \log(hT+1) + \left(\mu - \frac{h}{hT+1} - \frac{\sigma^2}{2}\right)t + \sigma W(t). \qquad (16)$$

Let $I = \log(a(0)) + \log(hT+1)$ and $\alpha = \frac{\mu}{\sigma} - \frac{h}{(hT+1)\sigma} - \frac{\sigma}{2}$. We further define $W_\alpha(t) = \frac{\log[c(t)] - I}{\sigma} = \left(\frac{\mu}{\sigma} - \frac{h}{(hT+1)\sigma} - \frac{\sigma}{2}\right)t + W(t)$ as the "discounted price," which follows Brownian motion at time zero with drift α. So the preseason cost $c(t)$ can be expressed as $\exp[g(0) + W_\alpha(t)\sigma]$, which is an increasing function of $W_\alpha(t)$.

To further our analysis, we derive from theorem 2.1 from Sakhanenko (2005) and we can get the following corollary.

Corollary 4.1. *The cumulative distribution of* $M_\alpha(t_1,t_2) := \min_{t_1\le u\le t_2} W_\alpha(u)$, *where* $W_\alpha(t) = \alpha t + W(t)$, *is given by*

$$P\left(M_\alpha(t_1,t_2) \le x + W_\alpha(t_1)\right) = \Phi\left(\frac{x-\alpha(t_2-t_1)}{\sqrt{t_2-t_1}}\right) + e^{2\alpha x}\Phi\left(\frac{x+\alpha(t_2-t_1)}{\sqrt{t_2-t_1}}\right),\ \alpha\in\mathbf{R},\ x\le 0)\ ,\ \textit{and}$$

$t \ge 0$, *where* $\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}\,\mathrm{dx}$.

Proof:

$$P(\min_{0\le u\le t}\left(W(u)+\alpha u\right)\le x)=P(-\min_{0\le u\le t}\left(W(u)+\alpha u\right)\ge -x)=P(\max_{0\le u\le t}\left(W(u)-\alpha u\right)\ge -x)$$

$$P(\min_{t_1\le u\le t_2}\left(W(u)+\alpha u\right)\le x)=P\left(\min_{0\le u\le t_2-t_1}\left(W(u)+\alpha u\right)\le x-W(t_1)-\alpha t_1\right).$$

Let $y=x+W(t_1)$, $M_\alpha(t_1,t_2):=\min_{t_1\le u\le t_2}W_\alpha(u)$, *and* $W_\alpha(t)=\alpha t+W(t)$, *then*

$$P\left(M_\alpha(t_1,t_2)\le x+W_\alpha(t_1)\right)$$

$$=P\left(\min_{0\le u\le t_2-t_1}\left(W(u)+\alpha u\right)\le x\right)$$

$$=P(\max_{0\le u\le t_2-t_1}\left(W(u)-\alpha u\right)\ge -x)$$

$$=\overline{\Phi}\left(\frac{-x+\alpha(\mathrm{t}_2-t_1)}{\sqrt{\mathrm{t}_2-t_1}}\right)+e^{2(-\alpha)(-x})\overline{\Phi}\left(\frac{-x-\alpha(\mathrm{t}_2-t_1)}{\sqrt{\mathrm{t}_2-t_1}}\right)$$

$$=\Phi\left(\frac{x-\alpha(\mathrm{t}_2-t_1)}{\sqrt{\mathrm{t}_2-t_1}}\right)+e^{2\alpha x}\Phi\left(\frac{x+\alpha(\mathrm{t}_2-t_1)}{\sqrt{\mathrm{t}_2-t_1}}\right)$$

This corollary provides us with the probability that the minimum value of the discounted price $W_\alpha(u)$ during $[i,T]$ is less than y; *i.e.,* the probability that level y has been reached by $W_\alpha(u)$ during $[i,T]$. Note that

$$p(\min_{i\le u\le T}W_\alpha(u)\le x+W_\alpha(i))$$

$$=p\left(\min_{i\le u\le T}\left[W_\alpha(u)\sigma+\log a(0)+\log(hT+1)\right]\le x\sigma+W_\alpha(i)\sigma+\log a(0)+\log(hT+1)\right)$$

$$=p\left(\min_{i\le u\le T}\log[c(u)]\le x\sigma+\log[c(i)]\right) \tag{17}$$

So at any time point i, $0\le i<T$, we can estimate the expected profit for purchasing with a certain preseason cost $c(i)e^{x\sigma}$ in the remaining periods $[i,T]$ as $E\left(\Pi^*\left[c(i)e^{x\sigma}\right]\right)P\left(M_\alpha(i,T)\le x\right)$. Furthermore, if $x=0$, $P\left(M_\alpha(i,T)\le x\right)$=1,

and $E\left(\Pi^*\left[c(i)e^{x\sigma}\right]\right)P\left(M_\alpha(i,T)\le x\right)=\Pi^*\left[c(i)\right]$, which is the profit at this i. As x decreases from zero, $E\left(\Pi^*\left[c(i)e^{x\sigma}\right]\right)$ increases from $E\left(\Pi^*\left[c(i)\right]\right)$ while $P\left(M_\alpha(i,T)\le x\right)$ decreases from 1. Therefore, if $\arg\max_x E\left(\Pi^*\left[c(i)e^{x\sigma}\right]\right)P\left(M_\alpha(i,T)\le x\right)<0$, we can get better-than-expected profit for purchasing at a lower preseason cost $c(i)e^{x\sigma}$ in the remaining periods $[i,T]$. Based on these observations, we can design our algorithm as follows.

<u>Algorithm for One Supplier</u>

$\Delta = T/n$, $t_i = i\Delta$
For $k=0,1,2,..(n-1)$
Do

Let $i = kT/n$, $c(i)=a(i)(hT+1)\left[\exp\left(\dfrac{-hi}{hT+1}\right)\right]$,

If $\arg\max_x E\left(\Pi^*\left[c(i)e^{x\sigma}\right]\right)P\left(M_\alpha(i,T)\le x\right)=0$

Then *Purchase at period i with* $y\left(p^*(c(i))\right)+z^*(c(i))$ *and stop.*

Else *Wait until the next period (set k=k+1).*

Loop

If $k=n$

Then *Purchase with* $y\left(p^*(c(T))\right)+z^*(c(T))$ *at time T.*

3.4. Properties and Lower Bound

From the previous section, we know that the optimal order quantity during the preseason purchasing and the optimal selling price during the selling season all depend on preseason cost. In the deterministic case, the retailer will increase the selling price p; in response to a higher preseason cost, the order quantity will be lowered, and the total profit will increase accordingly. We then show that such results also hold in the stochastic case studied in this chapter.

Proposition 4.1. *As preseason cost* c *increases, then*

i) The optimal selling price p^* *is strictly increasing in preseason cost* c

ii) The optimal order quantity q^* *is strictly decreasing in preseason cost* c *.*

iii) The optimal expected profit $E\left(\Pi[z^*(c), p^*(c), c]\right)$ *is strictly decreasing in preseason cost* c *.*

Proof:

Apply the result of Ha (2001) here, we can get *i*) and *ii*) of Proposition 4.1 directly.

Proof of iii: If $c^1 < c^2$, then

$$
\begin{aligned}
& E\left(\Pi[z^*(c^1), p^*(c^1), c^1]\right) \\
\geq\ & E\left(\Pi[z^*(c^2), p^*(c^2), c^2]\right) + E\left(\Pi[z^*(c^2), c^2, c^1]\right) \\
>\ & E\left(\Pi[z^*(c^2), p^*(c^2), c^2]\right).
\end{aligned}
$$

The lower the preseason cost, the higher the order quantity, the lower the selling price, and the higher the potential profit. Combined with Corollary 4.1, we can find out how those parameters affect the probability of reaching certain expected profits.

Proposition 4.2. *For a certain target expected profit* $E\left(\Pi^*\right)$*, which is higher than* $E\left(\Pi[z^*(c), p^*(c), c]\right)$ *where c is the present preseason cost, we can get:*

i) The longer the preseason period T, the higher the probability of reaching $E\left(\Pi^*\right)$ *in the preseason period.*

ii) The lower the preseason cost trend θ*, the higher the probability of reaching* $E\left(\Pi^*\right)$ *in the preseason period.*

iii) The lower the present preseason cost, the higher the probability of reaching $E(\Pi^*)$ *in the preseason period.*

Proof:

From the condition, we can get $x<0$, the

(i) $$\frac{\partial P(M_\alpha(0,T)\le x+W_\alpha(0))}{\partial T}$$

$$=\phi\left(\frac{x-\alpha T}{\sqrt{T}}\right)\left(-\frac{x}{2T^{3/2}}-\frac{\alpha}{2\sqrt{T}}\right)+e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)\left(-\frac{x}{2T^{3/2}}+\frac{\alpha}{2\sqrt{T}}\right)$$

$$=-\frac{x}{2T^{3/2}}\left[\phi\left(\frac{x-\alpha T}{\sqrt{T}}\right)+\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)\right]+\left(\frac{\alpha}{2\sqrt{T}}\right)\left[e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)-\phi\left(\frac{x-\alpha T}{\sqrt{T}}\right)\right] \quad . (18)$$

Here

$$\phi(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

As $x<0$, we get

$$-\frac{x}{2T^{3/2}}\left[\phi\left(\frac{x-\alpha T}{\sqrt{T}}\right)+\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)\right]>0\ . \tag{19}$$

If $\alpha>0$,

$$e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)<\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)<\phi\left(\frac{x-\alpha T}{\sqrt{T}}\right),$$

if $\alpha<0$,

$$e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)>\phi\left(\frac{x+\alpha T)}{\sqrt{T}}\right)>\phi\left(\frac{x-\alpha T}{\sqrt{T}}\right).$$

So

$$\left(\frac{\alpha}{2\sqrt{T}}\right)\left[e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)-\phi\left(\frac{x-\alpha T}{\sqrt{\mathrm{T}}}\right)\right]>0\,. \tag{20}$$

Combine (18) and (19), we get (17)>0. Then $P(M_\alpha(0,T)\leq x+W_\alpha(0))$ strictly increase in T.

So the longer the preseason period *T*, the higher the probability of the higher the probability of reaching Π^* in the preseason period.

(ii)

$$\frac{\partial P(M_\alpha(0,T)\leq x+W_\alpha(0))}{\partial\alpha}$$

$$=\sqrt{T}\left[e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)-\phi\left(\frac{x-\alpha T}{\sqrt{\mathrm{T}}}\right)\right]+2xe^{2\alpha x}\Phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right).$$

Because $2xe^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)$ *<0* and

$e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)=\frac{1}{\sqrt{2\pi}}e^{2\alpha x-\frac{x^2+2\alpha Tx+\alpha^2T^2}{2T}}=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2-2\alpha Tx+\alpha^2T^2}{2T}}=\phi\left(\frac{x-\alpha T}{\sqrt{\mathrm{T}}}\right)$, we can get

$\frac{\partial P(M_\alpha(0,T)\leq x+W_\alpha(0))}{\partial\alpha}$ *<0.*

iii) The lower the present preseason cost, the smaller *x* for a certain target expected profit.

$$\frac{\partial P(M_\alpha(0,T)\leq x+W_\alpha(0))}{\partial x}$$

$$=\frac{1}{\sqrt{T}}\left[e^{2\alpha x}\phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)-\phi\left(\frac{x-\alpha T}{\sqrt{\mathrm{T}}}\right)\right]+2\alpha\, e^{2\alpha x}\Phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)$$

$$=2\alpha\, e^{2\alpha x}\Phi\left(\frac{x+\alpha T)}{\sqrt{\mathrm{T}}}\right)>0.$$

This proposition accords with our intuition. When there is a longer time length to make the purchase before the selling season, there is more possibility of purchasing at a certain price and then gaining a target expected profit; when the preseason cost trend is lower, which may be because of the lower purchase price trend or because of the lower holding cost rate, there is a greater chance of reaching a certain profit in the preseason period; when the present preseason cost is lower, which is the result of a lower present purchase price, there is less chance to reach a target expected profit in the preseason period.

We pursue the lowest preseason cost, so that the expected profit can be maximized. However, because the price movement is stochastic, we can only try to get a low enough point, but it is almost impossible to catch the lowest preseason cost. Here we try to find the lower bound for the expected preseason cost so that we can find out how good our purchase cost is, and also estimate the lowest price we can get and its corresponding expected profit.

Proposition 4.3. *The expected lowest preseason cost in the preseason purchasing newsvendor problem is*

$$4(hT+1)\Phi\left(\frac{-2\theta T-\sigma}{2\sqrt{T}}\right)\exp\left(\frac{\sigma^2}{8T}+\frac{\sigma\theta}{2}\right)$$

$$+\frac{4(hT+1)\theta}{\sigma+2\theta}\Phi\left(\theta\sqrt{T}\right)-\frac{4(hT+1)\theta}{\sigma+2\theta}\exp\left[\frac{\sigma^2}{8T}+\frac{\theta^2+\vartheta\sigma}{2T}+\frac{2\theta^2+\theta\sigma}{2}\right]\Phi\left(\frac{-2\theta T-2\theta-\sigma}{2\sqrt{T}}\right)$$

.

Here $\theta=\frac{\mu}{\sigma}-\frac{h}{(hT+1)\sigma}-\frac{\sigma^2}{2}$.

Proof:

The preseason cost is $a(t)(hT+1-ht)$, here. $hT+1-ht\le(hT+1)\left[\exp\left(\frac{-ht}{hT+1}\right)\right]$.

$$a(t)(hT+1-ht)=(hT+1-ht)\exp\left((\mu-\frac{\sigma^2}{2})t+\sigma W(t)\right)$$

$$=(hT+1)\exp\left((\mu-\frac{h}{hT+1}-\frac{\sigma^2}{2})t+\sigma W(t)\right).$$

$$E\left\{\min_{0\le t\le T}\left[a(t)(hT+1-ht)\right]\right\}=E\left\{(hT+1)\exp\left[\min_{0\le t\le T}\left((\mu-\frac{h}{hT+1}-\frac{\sigma^2}{2})t+\sigma W(t)\right)\right]\right\}.$$

Let $\theta=\frac{\mu}{\sigma}-\frac{h}{(hT+1)\sigma}-\frac{\sigma^2}{2}$, then the above can be written as

$$(hT+1)E\left\{\exp\left[\min_{0\le t\le T}\left(\theta\sigma t+\sigma W(t)\right)\right]\right\}=(hT+1)a(0)\int_{-\infty}^{0}\exp(\sigma x)f_{\min\limits_{0\le t\le T}(\theta t+W(t))}(x)dx.$$

From Theorem 3.1 and Corollary 3.1, we get

$$f_{\min\limits_{0\le t\le T}(\theta t+W(t))}(x)=\begin{cases}\left(\frac{2}{\pi T}\right)^{1/2}\exp\left(-\frac{(x-\theta T)^2}{2T}\right)+2\theta\exp(2\theta x)\Phi\left(\frac{x+\theta T}{\sqrt{T}}\right), & x<0\\ 0, & x\ge 0\end{cases}$$

After solve this integration, we finally get

$$E\left\{\min_{0\le t\le T}\left[a(t)(hT+1-ht)\right]\right\}=4(hT+1)\Phi\left(\frac{-2\theta T-\sigma}{2\sqrt{T}}\right)\exp\left(\frac{\sigma^2}{8T}+\frac{\sigma\theta}{2}\right)+\frac{4(hT+1)\theta}{\sigma+2\theta}\Phi\left(\theta\sqrt{T}\right)$$

$$-\frac{4(hT+1)\theta}{\sigma+2\theta}\exp\left[\frac{\sigma^2}{8T}+\frac{\theta^2+\vartheta\sigma}{2T}+\frac{2\theta^2+\theta\sigma}{2}\right]\Phi\left(\frac{-2\theta T-2\theta-\sigma}{2\sqrt{T}}\right).$$

Proposition 4.4. The expected profit function $\Pi^*(c)$ is a convex function. $\frac{\partial^2\Pi(x)}{\partial x^2}\le 0$

Proof:

Assuming p^* and z^* are the optimal price p and demand variable part z for cost $tc_1 + (1-t)c_2$, p^{**} and z^{**} are the optimal p and z for cost c_1, p^{***} and z^{***} are the optimal p and z for cost c_2. Then we have

$$\Pi^*(tc_1 + (1-t)c_2) = \Pi(tc_1 + (1-t)c_2, p^*, z^*) = t\Pi(c_1, p^*, z^*) + (1-t)\Pi(c_2, p^*, z^*)$$

$$\leq t\Pi(c_1, p^{**}, z^{**}) + (1-t)\Pi(c_2, p^{***}, z^{***}) = t\Pi^*(c_1) + (1-t)\Pi^*(c_2)$$

So $\Pi^*(c)$ is a convex function.

Proposition 4.5. The expected profit in the preseason purchasing newsvendor problem is no lower than $\Pi^*(\hat{c})$,

where $\hat{c} = 4(hT+1)\Phi\left(\frac{-2\theta T - \sigma}{2\sqrt{T}}\right)\exp\left(\frac{\sigma^2}{8T} + \frac{\sigma\theta}{2}\right) + \frac{4(hT+1)\theta}{\sigma + 2\theta}\Phi\left(\theta\sqrt{T}\right)$

$$-\frac{4(hT+1)\theta}{\sigma+2\theta}\exp\left[\frac{\sigma^2}{8T} + \frac{\theta^2 + \vartheta\sigma}{2T} + \frac{2\theta^2 + \theta\sigma}{2}\right]\Phi\left(\frac{-2\theta T - 2\theta - \sigma}{2\sqrt{T}}\right)$$

and $\theta = \frac{\mu}{\sigma} - \frac{h}{(hT+1)\sigma} - \frac{\sigma^2}{2}$.

Here we assume the lowest cost during the time horizon *[0, T]* is represented by $c_{\min}$. Then for each possible price path, the corresponding profit is lower than $\Pi^*(c_{\min})$ (from Proposition 4.1), i.e., $\Pi^*(c) \leq \Pi^*(c_{\min})$. Then the corresponding expected profit for any strategy is no more than $E(\Pi^*(c_{\min}))$. Here $E(\Pi^*(c_{\min}))$ is the average profit if our purchasing price is always the lowest price at each possible price path. In Proposition 4.3, we prove that $E(c_{\min}) = \hat{c}$. Then by applying Taylor's approximation, we can prove that the expected profit at the lowest cost $c_{\min}$, i.e., $E\left(\Pi^*(c_{\min})\right)$, is lower than the profit at $E(c_{\min})$, i.e., $\Pi^*(\hat{c})$). The corresponding proof is as follows:

Proof:

From Proposition 4.4, the second derivative of Π*(c) is non-negative; therefore, Π*(c) is convex.

Use Taylor's approximation,

$$\Pi^*(c_{\min}) \approx \Pi^*(E(c_{\min})) + \frac{\partial \Pi^*(E(c_{\min}))}{\partial E(c_{\min})}\left(c_{\min} - E(c_{\min})\right) + \frac{1}{2}\frac{\partial^2 \Pi^*(E(c_{\min}))}{\partial E(c_{\min})^2}\operatorname{var}(c_{\min})$$

$$E(\Pi^*(c_{\min})) \approx \Pi^*(E(c_{\min})) + \frac{1}{2}\frac{\partial^2 \Pi^*(E(c_{\min}))}{\partial E(c_{\min})^2}\operatorname{var}(c_{\min}) \le \Pi^*(E(c_{\min})).$$

From Proposition 4.3,

$$E(c_{\min}) \le 4(hT+1)\Phi\left(\frac{-2\theta T - \sigma}{2\sqrt{T}}\right)\exp\left(\frac{\sigma^2}{8T} + \frac{\sigma\theta}{2}\right) + \frac{4(hT+1)\theta}{\sigma + 2\theta}\Phi\left(\theta\sqrt{T}\right)$$
$$- \frac{4(hT+1)\theta}{\sigma + 2\theta}\exp\left[\frac{\sigma^2}{8T} + \frac{\theta^2 + \vartheta\sigma}{2T} + \frac{2\theta^2 + \theta\sigma}{2}\right]\Phi\left(\frac{-2\theta T - 2\theta - \sigma}{2\sqrt{T}}\right).$$

Let $\hat{c} = 4(hT+1)\Phi\left(\frac{-2\theta T - \sigma}{2\sqrt{T}}\right)\exp\left(\frac{\sigma^2}{8T} + \frac{\sigma\theta}{2}\right) + \frac{4(hT+1)\theta}{\sigma + 2\theta}\Phi\left(\theta\sqrt{T}\right)$

$$- \frac{4(hT+1)\theta}{\sigma + 2\theta}\exp\left[\frac{\sigma^2}{8T} + \frac{\theta^2 + \vartheta\sigma}{2T} + \frac{2\theta^2 + \theta\sigma}{2}\right]\Phi\left(\frac{-2\theta T - 2\theta - \sigma}{2\sqrt{T}}\right).$$

Then $E(\Pi^*(c_{\min})) \le \Pi^*(\hat{c})$.

4. Extensions

4.1. Forward Contracts or Spot Market

In the above situation, we studied preseason purchasing from spot markets. To reduce the effect of price uncertainty, the retailer can sign a forward contract, in which the order price per unit is specified to hedge against the fluctuations of the purchase price. In a typical forward contract, the buyer makes a commitment to purchase from the supplier at a special time $\hat{t}$ with the contract unit price $\hat{a}$. To choose between a forward contract and the spot market, we need to compare the expected profits for purchasing from each of them.

First, the preseason cost using a forward contract can be calculated as $\hat{c} = \hat{a}[1 + h(T - \hat{t})]$. Subsequently, the corresponding optimal profit is $\Pi^*(\hat{c}) = E\left[\Pi\left(z^*(\hat{c}), p^*(\hat{c}), \hat{c}\right)\right]$. Next, let us look at the spot market. As we discussed in the previous section, the optimal profit can be roughly estimated using $\max_x \left\{\Pi^*\left[\exp\left(g(t_1) + x\sigma\right)\right] P\left(M_\alpha(t_1, T) - W_\alpha(t_1) \le x\right)\right\}$ (for a more accurate result, we need to run large numbers of simulations and find the corresponding average profit using the methods introduced in the previous sections as the expected profit from the spot market). So from the profit-maximization aspect, if $\Pi^*(\hat{c}) \ge \max_x \left\{\Pi^*\left[\exp\left(g(t_1) + x\sigma\right)\right] P\left(M_\alpha(t_1, T) - W_\alpha(t_1) \le x\right)\right\}$, we should choose the forward contract. The corresponding optimal order quantity, selling price, and profit can be calculated using (7) and (8); otherwise, we should consider purchasing from the spot market.

4.2. Purchasing from Multiple Suppliers

In this section, we examine how to choose among multiple suppliers to maximize profit. We assume there are n independent suppliers and the price of supplier k at time t is a process $a_k = \{a_k(t), t \ge 0\}$, which is expressed by the stochastic differential equation $\mathrm{d}a_k(t) = a_k(t)\{\mu_k \,\mathrm{d}t + \sigma_k \,\mathrm{d}W_k(t)\}$. Correspondingly, we define $\alpha_k = \frac{\mu_k}{\sigma_k} - \frac{h_k}{(h_k T + 1)\sigma_k} - \frac{\sigma_k}{2}$, $W_{\alpha_k}(t) = \left(\frac{\mu_k}{\sigma_k} - \frac{h_k}{(h_k T + 1)\sigma_k} - \frac{\sigma_k}{2}\right)t + W_k(t)$, and $M_{\alpha_k}(t_1, t_2) := \min_{t_1 \le u \le t_2} W_{\alpha_k}(u)$. As in the single supplier case, we first divide the time interval $[0, T]$ into n small observation time periods. $\Delta = T / n$, and the unit purchasing cost $a_k(t)$ represents the unit purchasing cost of the t^{th} period, *i.e.*, the unit purchasing cost at time point $t\Delta$. At each new period t, we first choose the highest present profit among all the suppliers, *i.e.*, $\max_k \Pi^*[c_k(t)]$, where $c_k(t) = a_k(t)(1 + hT - ht)$ represents the unit cost of supplier *k*, and $j = \arg\max_k \Pi^*[c_k(i)]$ represents the supplier with the highest expected profit at present preseason cost c_k, $k = 1,2...,n.$. Then we calculate the highest expected target profit that at least one supplier can reach in the remaining time $[t\Delta, T]$, which equals the product of the target profit and the probability of any of the suppliers' prices reaching this target expected profit. The corresponding target expected profit can then be described as:

$$\arg\max_x \Pi^*\left[c_j(t)e^{x\sigma}\right]\left[1 - \Pi_{k=1}^n P\left(M_{\alpha_k}(t, T) > \frac{\log[c_j(t)] - \log[c_k(t)]}{\sigma_k} + x\right)\right].$$

If the target expected profit is no higher than the highest present profit, *i.e.*,

$$\arg\max_x \Pi^*\left[c_j(t)e^{x\sigma}\right]\left[1 - \Pi_{k=1}^n P\left(M_{\alpha_k}(t, T) > \frac{\log[c_j(t)] - \log[c_k(t)]}{\sigma_k} + x\right)\right] = 0,$$

we will purchase from supplier j right away; otherwise, we will wait till the next period and continue in this manner until the units have been purchased or the end of the time horizon has been reached. Because

$$1-\Pi_{k=1}^{n}P\left(M_{\alpha_k(t)}(t,T)>\frac{\log[c_j(t)]-\log[c_k(t)]}{\sigma_k}+x\right)\geq \max_{1\leq k\leq n}P\left(M_{\alpha_k(t)}(t,T)<\frac{\log[c_j(t)]-\log[c_k(t)]}{\sigma_k}+x\right),$$

purchasing from multiple suppliers provides a better expected profit than solely purchasing from any of the suppliers individually. Thus, the expected profit increases along with the number of suppliers *n*.

4.3. Risk Minimization

In the previous part, we discussed how to compare the profit of purchasing from the spot market with purchasing from a forward contract. An important purpose of a forward contract is to hedge against risk. As the purchase price has been set in the forward contract, the risk of the fluctuation of the spot market price is avoided. So whether or not we should choose purchasing from the spot market or purchasing from the forward contract, we should not only compare their profits, but also consider the risks in the spot market.

Similar to Gan, Sethi, and Yan (2005), we consider the risk of using a certain supplier as the probability that the realized profit is less than or equal to the retailer's specified target unit profit. Let γ be the target unit profit. Clearly, the optimal strategy to minimize risk is to purchase when the expected profit is bigger than γ during [0, *T*]. As we discussed in previous sections, $\Pi^*(c)$ has a unique solution for each preseason unit cost c, so for each target profit γ, we can get a benchmark preseason unit cost of $\tilde{c}=\Pi^{*-1}(\gamma)$. Then the risk of choosing suppliers 1 to *n* at target unit profit γ from time *t* to time *T* is

$$\left[1-\Pi_{k=1}^{n}P\left(M_{\alpha_k}(t,T)>\frac{\log\left[\Pi^{*-1}(\gamma)\right]-\log[c_k(t)]}{\sigma_k}\right)\right] \tag{21}$$

To minimize the risk expressed in (18), the corresponding optimal purchasing strategy is to purchase right away from supplier *k* when its preseason unit cost drops below $\tilde{c}$, *k*=1,2…,*n*.

When we calculate the risk of our purchasing strategy introduced in section 3.2.2, the profit can be estimated using $\gamma = \Pi^*(\hat{c})$ .So the corresponding risk to purchase from *1, 2* …or *n* suppliers will be $\left[1-\Pi_{k=1}^{n} P\left(M_{\alpha_k}(t,T) > \frac{\log[\hat{c}] - \log[c_k(t)]}{\sigma_k}\right)\right]$.

4.4. Multiple Demand Points

A natural extension of preseason purchasing and in season selling problem studied in the previous section is a corresponding management situation involving multiple demand points, where units are stocked to satisfied demands in multiple time points. Let $D_j(p,\varepsilon) = y(p) + \varepsilon_j$ be the demand of the final product at time t_j, $j = 1,2...,m$, where $t_1 < t_2 < ... < t_m$, units left over from one time point will not be available to meet demand in subsequent time points, $y(p)$ is a downward sloping, concave, deterministic function of the unit selling price, and ε_j has a linear or log-concave density function $f(x)$ with mean μ and variance σ^2 . Thus, for $j = 1,2...,m$, the D_j is needed to be stocked before t_j . When setup cost for each time is zero, obviously, such a problem can be decomposed into m subproblems in which the j^{th} problem has a time period $[0,t_j]$ to make the purchase to satisfy the demand at time t_j . When setup cost is not zero, such a problem can be decomposed into m subproblems in which the j^{th} problem compare the expected profit of having a time period $[0,t_j]$ to make the purchase to satisfy demand D_j and the expected profit of having a time period $[0, t_{j-1}]$ to make the purchase to satisfy demand $D_j + D_{j-1}$ minus the setup cost and then choose the one with higher profit.

When units left over from one time point will be available to meet demand in subsequent time points, this problem will be more complicated. A special case is that there exists a salvage market such that each leftover remaining at the end of a period can be sold at the corresponding preseason cost. Thus, the multiple-period problem reduces to a sequence of single-period problems as no left over being carried to the next period case. People who interest in this question can refer to Petruzzi and Dada (1999) for detail discussion.

4.5. Presale Procurement and Production Problem

If the firm manufactures the product from the distinct raw material from the market before the selling season and sell to the market during the selling season, in addition to purchasing cost and holding cost, there are additional production time and production cost. Referring to the Chen and Munson (2004) model, we assume k is the unit production cost if the production time length equal or bigger than $\widetilde{t}$,where $0 < \widetilde{t} \leq T$ .Then the unit production cost $G(t) = k + M(t)$, where $M(t)$ is a multiplier function that is assumed to be concave and strictly decreasing with respect to production run length t when $0 < t \leq \widetilde{t}$ and $M(t) = 0$ when $t > \widetilde{t}$ .As a special case, the multiplier function could take the exponential form of $M(t) = e^{-\lambda t} I(t \leq \widetilde{t})$.

If we purchase at time $t \in [0,T]$, then the unit holding cost before the selling season is $a(t)[h(T-t)]$ and the unit production cost is $k + M(T-t)$. The unit purchasing, holding, and manufacturing cost before selling season is $c = c(t) = a(t) + a(t)[h(T-t)] + k + M(T-t)$.

The backward deduction Binomial Tree method is applied as follows: First the time interval $[0,T]$ is divided into n small time intervals. $\Delta = T/n$, the unit presale cost $C(t_1 + (t+1)\Delta, t_1)$, which represent the unit purchasing cost of the ith node of the $(t+1)\Delta$ time interval of the binomial tree satisfies:

$$C_i[t_1 + (t+1)\Delta, t_1] = \begin{cases} C_i(t_1 + t\Delta, t_1)e^{\sigma\sqrt{\Delta}} & prob \quad \varsigma = \dfrac{1}{2}\left(1 + \dfrac{\mu - .5\sigma^2}{\sigma}\sqrt{\Delta}\right) \\ C_{i+1}(t_1 + t\Delta, t_1)e^{-\sigma\sqrt{\Delta}} & prob \quad 1-\varsigma = \dfrac{1}{2}\left(1 - \dfrac{\mu - .5\sigma^2}{\sigma}\sqrt{\Delta}\right) \end{cases}$$

Then the profit is $\{k + M[T-(t+1)\Delta]\}U_1(0)$. The corresponding recursively backward is: $U_i(t\Delta) = \max\left\{\Pi^*\left[a_i(t\Delta)(1 + hT - ht\Delta) + k + M(T - t\Delta)\right], \varsigma U_i[(t+1)\Delta] + (1-\varsigma)U_{i+1}[(t+1)\Delta]\right\}$, then the decision function is as follows:

If $\Pi^*\left[a_i(t\Delta)(1 + hT - ht\Delta) + k + M(T - t\Delta)\right] \geq \varsigma U_i[(t+1)\Delta] + (1-\varsigma)U_{i+1}[(t+1)\Delta]$, the optimal decision at this node is to purchase with order quantity q^* immediately, otherwise the optimal decision is to wait for another period. The corresponding order quantity q^* and the potential selling price p^* are calculated the same as the ones in preseason purchasing part:

The approximate method is as follows:

We can approximate the calculation by finding $\tau \leq 0$ and η that maximize $\Lambda(\tau,\eta)$,

$\Lambda(\tau,\eta) = \sum_{t=1}^{T}\left\{\Pi^*\left[a_i(t\Delta)(1 + hT - ht\Delta) + k + M(T - t\Delta)\right]\left[F_{\varphi(\tau+\eta t)}(t) - F_{\varphi(\tau+\eta t)}(t-1)\right]\right\}$. $\Lambda(\tau,\eta)$ has a unique optimal solution, where τ^* is the unique τ that satisfies $d\Lambda(\tau,\eta(\tau))=0$ and

$\eta^* = \eta(\tau^*)$.

We then approximate the calculation by finding $\tau \le 0$ and η that maximize

$\sum_{t=1}^{T}\{\Pi^*[a_i(t\Delta)(1+hT-ht\Delta)+k+M(T-t\Delta)][F_{\varphi(\tau+\eta t)}(t)-F_{\varphi(\tau+\eta t)}(t-1)]\}$.If we use $\Lambda_2(\tau,\eta)$ to

represent this formula, similar to proposition 3.4, $\Lambda(\tau,\eta)$ has a unique optimal solution,

where τ^* is the unique τ that satisfies $d\Lambda(\tau,\eta(\tau))=0$ and $\eta^* = \eta(\tau^*)$. If $\tau = 0$, it means the

present purchasing is better than later purchasing and the purchasing strategy is purchasing right away. The corresponding order quantity and selling price are:

$c = a_i(t\Delta)(1+hT-ht\Delta)+k+M(T-t\Delta)$, $z(p\,|\,c) = F^{-1}\left(\dfrac{p+s-c}{p+s+v}\right)$,

$p^*(c) = \left\{p \,|\, \dfrac{\partial E[\Pi(p, z(p\,|\,c))]}{\partial p} = 0\right\}$, and the corresponding expected profit is

$\Pi^*(c) = \Pi(z^*(c), p^*(c), c)$.

4.6. Make a Second Purchase

If we omit the setup cost for each purchase, multiple times purchasing can provide no worse profit than one time purchasing and the expected profit will increase as the number of purchase events increases. One time purchasing can be seen as a special case of multiple purchasing in which the second and up purchasing have zero order quantities. Finding the optimal purchasing strategy for multiple times purchasing is very difficult and may vary in different scenarios. However, even we can not find an optimal purchasing strategy for multiple times purchasing, we can design a certain purchasing strategy to improve the expected profit. For example, if we have made a purchasing at time $t < T$ with a certain preseason cost $\hat{c}$, and we make a second purchasing during $[t,T]$ if and only if the preseason cost is lower than $\hat{c}$. The probability that we make a second purchasing equals to the probability of the cost is lower than $\hat{c}$ during $[t,T]$ and the corresponding preseason cost drop from $\hat{c}$ to $e^{-x}\hat{c}$

with $x > 0$ is $P(M_{\alpha}(t,T) \le x) = \Phi\left(\frac{-x-\alpha(\mathrm{T}-t_1)}{\sqrt{\mathrm{T}-t}}\right) + e^{-2\alpha x}\Phi\left(\frac{-x+\alpha(\mathrm{T}-t)}{\sqrt{\mathrm{T}-t}}\right)$. Because the cost of the second purchase is lower than the first one. So the average cost of both the first and second purchasing is lowered. So according to propositions 4.1 and 4.2, the optimal order quantity will increase and the optimal profit will increase too. In this way, a second purchasing will increase the expected profit than single purchasing for sure.

We assume that we have made a purchasing at time $t_1 < T$ with the corresponding purchasing price $a(t_1)$, order quantity q_1 and potential selling price p_1, then the preseason cost $c_1 = a(t_1)(hT+1)\left[\exp\left(\frac{-ht_1}{hT+1}\right)\right]$ and the corresponding potential profit is $\Pi[p_1, q_1 \mid c_1]$. We assume that for each purchasing, there is a setup cost M, $M \ge 0$. When the preseason cost is c, $\Pi_2[p,q \mid c]$ is to find the optimal order quantity and optimal selling price for the total order quantities from both orders so that the increase of profit is maximized. The expected profit for the second purchasing can be expressed as

$$E[\Pi_2(z,p \mid c)] = \int_{-\infty}^{z} \left(p[y(p)+u+q_1] - v[z-u]\right)\phi(u)du$$

$$+ \int_{z}^{+\infty} \left(p[y(p)+z+q_1] - s[u-z]\right)\phi(u)du - c[y(p)+z] - c_1 q_1 - a\Pi[p_1, q_1 \mid c_1] - M$$

Then we can derive from the above equation and calculate the optimal price and order quantity as follows:

$$p(z \mid c) = \frac{g + bc + \mu + q_1 - \int_z^{+\infty} (u - z) f(u) du}{2b} \tag{22}$$

$$z^*(c) = \left\{ z \mid \frac{\partial E[\Pi(z, p(z \mid c)) \mid c]}{\partial z} = 0 \right\} \tag{23}$$

$$\frac{\partial E[\Pi_2(z, p(z \mid c)) \mid c]}{\partial z} = \Phi(z) \left(\frac{\int_z^{+\infty} (u - z)\phi(u) du + \mu - a}{2b} - \frac{c}{2} - v - s \right)$$

$$- \frac{\int_z^{+\infty} (u - z)\phi(u) du + \mu + a}{2b} - \frac{c}{2} + s = 0$$

It is easy to derive in a similar way as proposition 4.1 that the corresponding $z^*(c)$ and $p^*(c) = p(z^*(c) \mid c)$ are unique. The corresponding optimal profit as the preseason cost at c is $\Pi_2[z^*(c), p^*(c), c]$, and we use $\Pi_2^*(c)$ to represent it.

When now we are at time t_2, where $t_1 < t_2 \le T$, the purchasing price is $a(t_2)$ and a second purchasing right away is profitable if and only if profit $\Pi_2^*[a(t_1)(hT + 1 - ht_1)] > 0$ and order quantity $z + y[p^*(c)] > 0$.

The backward deduction Binomial Tree method is applied for time interval $[t_1, T]$ as follows: First the time interval $[t_1, T]$ is divided into n small time intervals. $\Delta = (T - t_1)/n$, the unit presale cost $C(t_1 + (t+1)\Delta, t_1)$, which represent the unit purchasing cost of the ith node of the $(t+1)\Delta$ time interval of the binomial tree satisfies:

$$C_i[t_1+(t+1)\Delta,t_1]=\begin{cases} C_i(t_1+t\Delta,t_1)e^{\sigma\sqrt{\Delta}} & prob \quad \varsigma=\frac{1}{2}\left(1+\frac{\mu-.5\sigma^2}{\sigma}\sqrt{\Delta}\right) \\ C_{i+1}(t_1+t\Delta,t_1)e^{-\sigma\sqrt{\Delta}} & prob \quad 1-\varsigma=\frac{1}{2}\left(1-\frac{\mu-.5\sigma^2}{\sigma}\sqrt{\Delta}\right) \end{cases}$$

The corresponding recursively backward is:

$$U_i(t\Delta)=\max\left\{\Pi_2^*\left[a(t\Delta)(hT+1)\left[\exp\left(\frac{-ht\Delta}{hT+1}\right)\right]\right],\varsigma U_i[(t+1)\Delta]+(1-\varsigma)U_{i+1}(t+1)\Delta],0\right\}$$

Then the decision rule is as follows:

If $U_i(t\Delta)=\Pi_2^*\left[a(t\Delta)(hT+1)\left[\exp\left(\frac{-ht\Delta}{hT+1}\right)\right]\right]$, the optimal decision at this node is to purchase with order quantity $q^*\left[a(t\Delta)(hT+1)\left[\exp\left(\frac{-ht\Delta}{hT+1}\right)\right]\right]$ immediately, otherwise the optimal decision is to wait for another period.

For the above solution process, we can also derive a quick solution heuristic. First we use the one time purchasing strategy to make the first purchasing at t_1, then after the first purchasing, we purchase at a specific cost level which is decided by comparing different exponential cost curves and choose the one that maximize the profit , *i.e.*, finding the exponential cost curve in which $\tau\le 0$ and η that maximize

$$\sum_{t=1}^{T}\left\{\Pi_2^*\left[a(t)(hT+1)\left[\exp\left(\frac{-ht}{hT+1}\right)\right]\right]\left[F_{\varphi(\tau+\eta t)}(t)-F_{\varphi(\tau+\eta t)}(t-1)\right]\right\}.$$

If $\tau=0$, it means the present purchasing is better than later purchasing and the purchasing strategy is purchasing right away. Here $c=a(t\Delta)(hT+1)\left[\exp\left(\frac{-ht\Delta}{hT+1}\right)\right]$, then the corresponding order quantity and selling price can be calculated using (22) and (23) and

the corresponding expected profit is $\Pi_2^*\left[a(t\Delta)(hT+1)\left[\exp\left(\frac{-ht\Delta}{hT+1}\right)\right]\right]$.

5. Numerical Analysis

5.1. Profit Impact of Increasing Cost Trends

CASE I: *Demand* $=150-\frac{1}{3}$*selling price* $+\varepsilon$, *ε is standard normal. For the purchasing price movement, we let* $\sigma=1$, *T = 100* weeks, $k=150$ *and* $h=.007$ *purchase price / week* .

In this example, we define $y(p)=g-bp$ and assume that ε follows a standard normal distribution. We also let the disposal cost v and shortage cost s be zero. Then equation (10) can be written as $z^*(p\,|\,c)=\Phi^{-1}\left(\frac{p-c}{p}\right)$ and the optimal selling price $p^*(c)$ by solving this

equation $g-2bp+bc+\frac{c}{p}\Phi^{-1}\left(\frac{p-c}{p}\right)+\int_{-\infty}^{\phi^{-1}(\frac{p-c}{p})}x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=0$. Then $q^*=z^*(p^*\,|\,c)+g-bp$.

The corresponding profit can be simplified

as $E[\Pi(z^*,p^*\,|\,c)]=(p^*-c)q^*-p^*\int_{-\infty}^{z^*}(z^*-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$. From previous analysis, fixing some parameters of the selling season will not influence the properties of these analyses and can be easily extended to other cases.

In this case, we run the simulation 100 times and calculated the averages of the simulation results and the present expected profit is 356.91.

Table 4 compares the results of “purchasing at the highest expected profit” with “purchasing at the lowest expected cost”. The averages for the purchasing time, order quantity, selling price, and expected profit of “purchasing at the highest expected profit” method are on the left side, and the averages for the purchasing time, order quantity, selling price, and expected profit of "purchasing at the lowest expected cost" method are on the right side. Through the table, we notice the big differences of the expected profit of these two methods. The expected profit of "purchasing at the highest expected profit" is about 21.7% higher than the expected profit of "purchasing at the lowest expected cost" at $\mu = .05$. So "purchasing at the lowest expected cost" may lead to much lower expected profit than the "purchasing at the highest expected profit" method when $\theta > -\sigma / 2$, i.e., the cost trend μ is more than holding cost percentage h. If $\theta < -\sigma / 2$, *i.e.*, the purchasing cost is increasing very slowly and is lower than the holding cost, the two methods comes to the same results and that is "purchasing at the end". However, for many commodities like rice, oil, gas, and fuel, the high cost increasing trends are much more common.

Table 4 also provides averages for the purchasing time, order quantity, selling price, and expected profit during time period $[0,100]$ using our strategy when purchasing cost trends μ changes from .05 to .7. From the table, when μ is no more than .62, the higher μ is, the lower will be the expected profit and order quantity, and the higher the potential selling price will be. This result is consistent with proposition 4.2. The ideal purchasing strategy at time 0 is to wait for some time instead of purchasing right away. When μ is more than .62, it is better to purchase right away and the corresponding expected profit is \$356.91. So for a slightly positive cost trend, even if the cost is increasing, it is still possible to gain better profit by waiting for some time to catch the chance to purchase at a lower cost from the fluctuating market. As μ increases, the profit increases but at a decreasing rate, and the purchasing time also increases at a decreasing rate, *i.e.,* we can wait for a short time but we'd better not purchase too late. When the purchasing cost trend is high enough, we'd better purchase at the very beginning.

Table 4. Profit Comparison for Changing Purchasing Cost Drift Term

	Solved by Maximizing Expected Profit				Solved by Minimizing Expected Cost			
μ	Order Quant	Price	Purchasing. Time	Profit	Order Quant	Price	Purchasing. Time	Profit
0.05	7.4	370	28.94	434.5	3.43	377.8	0	356.91
0.1	7.13	370	26.67	434.5	3.43	377.8	0	356.91
0.15	6.75	370	26.15	434.5	3.43	377.8	0	356.91
0.2	6.41	370	25.21	434.48	3.43	377.8	0	356.91
0.25	6.1	370	24.21	434.46	3.43	377.8	0	356.91
0.3	5.71	370.04	23.46	434.08	3.43	377.8	0	356.91
0.36	5.31	370.08	21.76	433.56	3.43	377.8	0	356.91
0.4	5.02	370.19	21.57	432.21	3.43	377.8	0	356.91
0.44	4.8	370.37	18.78	430.23	3.43	377.8	0	356.91
0.46	4.61	370.9	18.74	426.13	3.43	377.8	0	356.91
0.5	4.38	371.14	17.96	422.06	3.43	377.8	0	356.91
0.54	4.2	371.64	14.21	416.69	3.43	377.8	0	356.91
0.58	3.98	372.66	8.33	406.24	3.43	377.8	0	356.91
0.6	3.88	373.33	7.57	399.67	3.43	377.8	0	356.91
0.62	3.74	374.57	6.54	387.64	3.43	377.8	0	356.91
0.64	3.43	377.8	0	356.91	3.43	377.8	0	356.91
0.66	3.43	377.8	0	356.91	3.43	377.8	0	356.91
0.7	3.43	377.8	0	356.91	3.43	377.8	0	356.91

Figure 9 further illustrates the corresponding profit increase trends when μ increased from .05 to .7 and figure 10 illustrates the corresponding purchase time trends. From figure 9 and 10, we observe that as t μ increases, the profit increases but at a decreasing rate and the purchasing time also increases at a decreasing rate.

Figure 9. Numerical Example of Profit Trend when Mu Increases

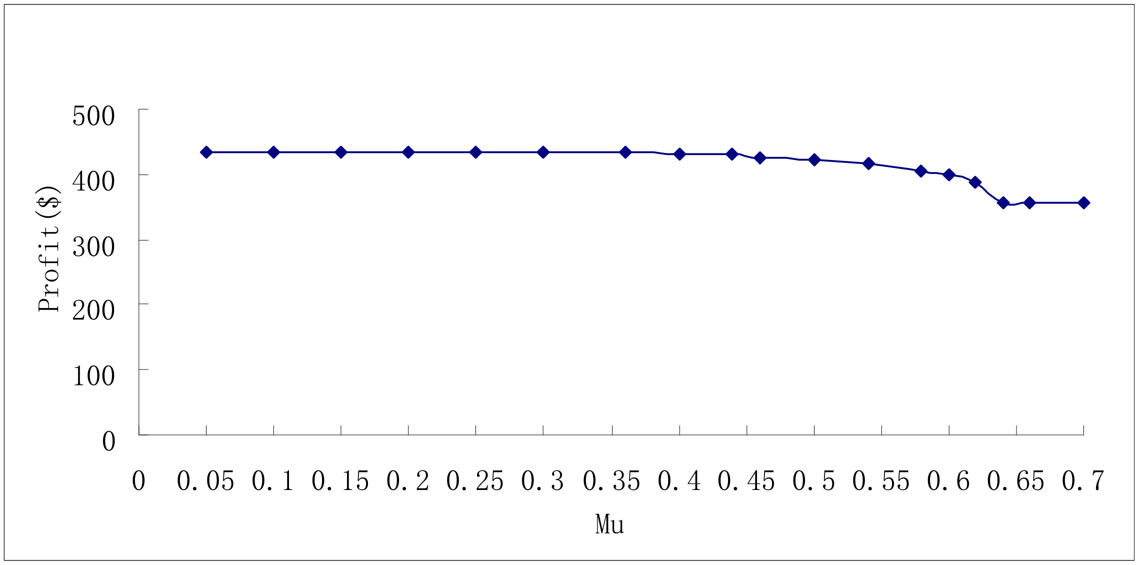
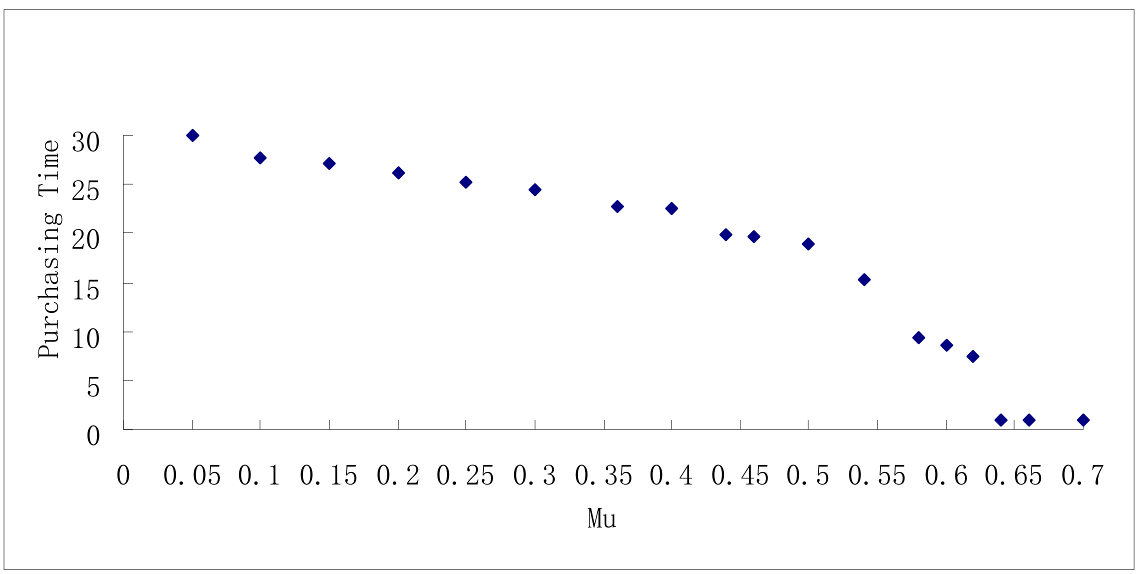


Figure 10. Numerical Example of Purchasing Time Trend when Mu Increases



CASE II: $Demand = 3.7 - .005 selling\ price + \varepsilon$, *$\varepsilon$ is truncate normal defining within* $(0,+\infty)$ *with mean 0 and standard deviation 2. For the purchasing price movement, let* $\mu = .06$, $\sigma = 1$, $T = 20$ weeks.

In this example, we examine how the holding cost percentage impacts the purchasing decision and expected profit.

We define $y(p) = g - bp$ and assume that ε follows a truncated normal distribution within the interval $(0,+\infty)$ with mean 0 and variance 1, *i.e.*, $f(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}, x \in (0,+\infty)$ .We also let the dispose cost v and shortage cost s be zero. Then equation (10) can be written as $z^*(p|c) = \Phi^{-1}\left(\frac{2p-c}{2p}\right)$ and (11) can be simplified as

$g - 2bp + bc + \frac{c}{p}\Phi^{-1}\left(\frac{2p-c}{2p}\right) + \int_0^{\phi^{-1}(\frac{2p-c}{2p})} x \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$. We can get the optimal selling price $p^*(c)$ by solving this equation, and then $q^* = z^*(p^*|c) + g - bp$. The corresponding profit can be simplified as $E[\Pi(z^*, p^*|c)] = (p^* - c)q^* - p^* \int_0^{z^*} (z^* - x) \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$.

We let $g = 3.7$, b=.005, h=.1, and the present cost $c = 20.09$.

Table 5 compares the results of "purchasing at the highest expected profit" with "purchasing at the lowest expected cost". The averages for the purchasing time, order quantity, selling price, and expected profit of "purchasing at the highest expected profit" method are on the left side, and the averages for the purchasing time, order quantity, selling price, and expected profit of "purchasing at the lowest expected cost" method are on the right side. In the table, the expected profit of "purchasing at the highest expected profit" is higher than "purchasing at the lowest cost" when purchasing cost trend μ is between .14 and .54 . When μ is .14, the expected profit of "purchasing at the highest expected profit" is about 7.8% higher than the expected profit of "purchasing at the lowest expected cost". So "purchasing at the lowest expected cost" may lead to much lower expected profit than the "purchasing at the highest expected profit" method. When $\theta > -\sigma/2$, i.e., the cost trend μ is more than holding cost parameter h in this case, our purchasing depends on how much is the difference of μ and h. If μ is much higher than h, we'd better purchase at the very beginning, otherwise, wait a little bit later to make a purchase. If $\theta < -\sigma/2$, i.e., the purchasing cost is increasing very slowly and is lower than the holding cost, the two methods come to the same results and that is "purchasing at the end". However, today, high increasing cost trends are much more common.

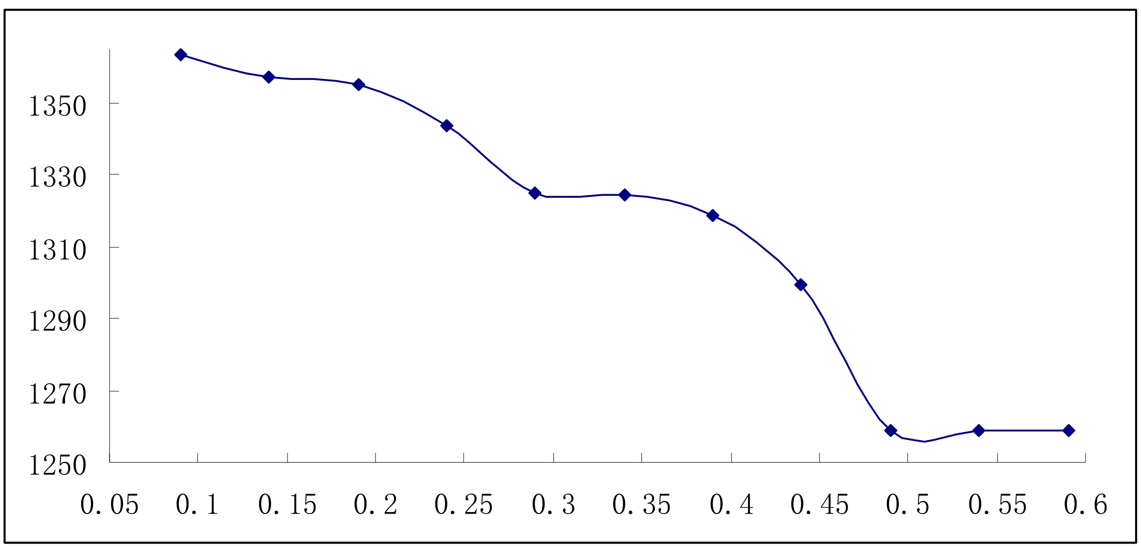
The following table also provides averages for the purchasing time, order quantity, selling price, and expected profit during time period $[0,100]$ using our strategy when the cost trend changes from .09 to .59. From the table, when μ is no more than .1, the higher μ is, the lower will be the expected profit and order quantity, and the higher the potential selling price will be. The ideal purchasing strategy at time 0 is to wait for some time instead of purchasing right away. When μ is more than .49, it is better to purchase right away and the corresponding expected profit is $466.9. So for a slightly positive cost trend, even if the cost is increasing, it is still possible to gain better profit by waiting for some time to catch the chance to purchase at a lower cost from the fluctuating market. As μ increases, the profit increases, but the purchasing time increases at a decreasing rate, *i.e.,* we can wait for a short time but we'd better not purchase too late.

Figure 11 further illustrates the corresponding profit increase trend when μ increased from .09 to .59, we observe that as μ increases, the profit decreases but the decreasing rate is not so smooth. Sometime it decreases fast and sometimes it decreases slowly. However our sample size is 100, but the potential paths are countless, so the plot can only describe the rough trend of expected profit.

Table 5. Profit Comparison for Changing Holding Cost

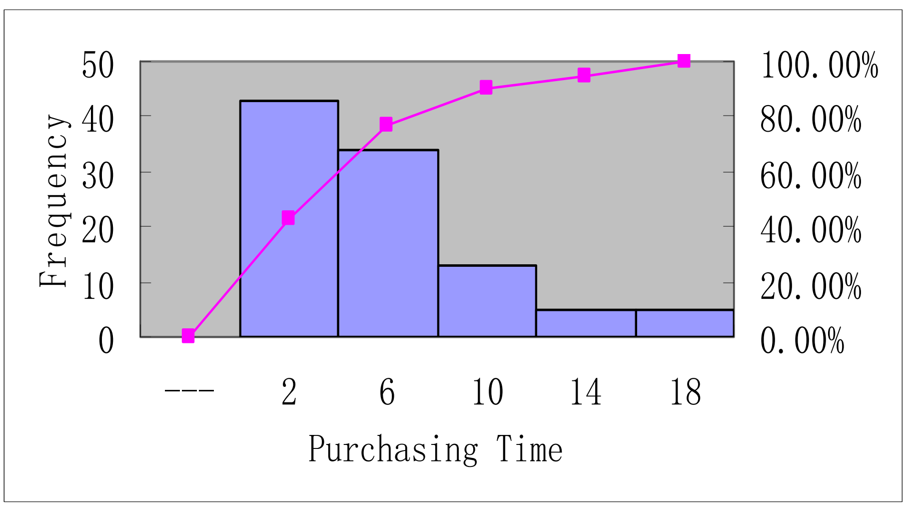
μ	Maximize Expected Profit					Minimize Expected Cost				
	Avg. Profit	**Avg. Quant**	**Avg Price**	**Avg Time**	**Avg Cost**	**Avg Profit**	**Avg Quant.**	**Avg Price**	**Avg Time**	**Avg Cost**
0.09	1364	7.867	450.9	8.78	7.651	1366	11.44	450.5	20	7.833
0.14	1358	7.483	451.8	7.43	2.019	1259	5.412	466.9	0	20.09
0.19	1355	7.268	452.1	6.22	6.751	1259	5.412	466.9	0	20.09
0.24	1344	6.872	453.9	6.06	298	1259	5.412	466.9	0	20.09
0.29	1325	6.52	456.8	6.47	10.44	1259	5.412	466.9	0	20.09
0.34	1324	6.431	456.4	5.52	50.07	1259	5.412	466.9	0	20.09
0.39	1319	6.257	457.8	4.41	46.68	1259	5.412	466.9	0	20.09
0.44	1299	5.968	460.8	4.28	157.9	1259	5.412	466.9	0	20.09
0.49	1259	5.412	466.9	0	20.09	1259	5.412	466.9	0	20.09
0.54	1259	5.412	466.9	0	20.09	1259	5.412	466.9	0	20.09
0.59	1259	5.412	466.9	0	20.09	1259	5.412	466.9	0	20.09

Figure 11. Numerical Example of Profit Trend when Mu Increases



To show how the purchasing works, we choose one example that $\mu = .29$, in which the purchasing is not at the very beginning. We have run the simulation 100 times. The purchasing price movement satisfies the Black-Scholes equation and in each time the path is different, so do our purchasing time, order quantity, and pricing strategies and the corresponding profit. Figure 12 and Figure 13 show how the purchasing time distributes and how the corresponding expected profit distributes in these 100 times simulation. Figure 12 shows that the purchasing time more falls in the early stage and the chances of purchasing drops as time moves on. Only about ten times the purchasing is made after 12 weeks. Figure 13 shows that the expected profit distributes from 950 to 1400, but it more focuses on the range of 1300 to 1400 and only about 15%, the expected profit is below 1300.

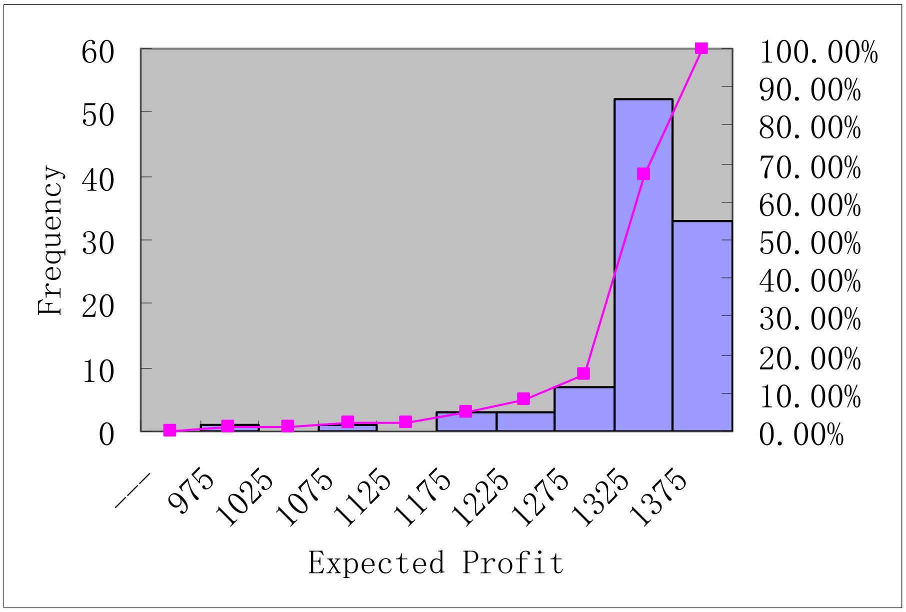
Figure 12. Numerical Example of the Distribution of Purchasing Time



The Summary Table

Range	Frequency	Percentage	Cumulative %
[0,4)	43	43.00%	43.00%
[4,8)	34	34.00%	77.00%
[8,12)	13	13.00%	90.00%
[12,16)	5	5.00%	95.00%
[16,20)	5	5.00%	100.00%

Figure 13. Numerical Example of the Distribution of Expected Profits



The Summary Table

Range	Frequency	Percentage	Cumulative %
[950,1000)	1	1.00%	1.00%
[1000,1050)	0	0.00%	1.00%
[1050,1100)	1	1.00%	2.00%
[1100,1150)	0	0.00%	2.00%
[1150,1200)	3	3.00%	5.00%
[1200,1250)	3	3.00%	8.00%
[1250,1300)	7	7.00%	15.00%
[1300,1350)	52	52.00%	67.00%
[1350,1400)	33	33.00%	100.00%

CASE III: *Demand* $= 400 - 5$*selling price* $+ \varepsilon$, ε *is truncate normal defining within* $(0,+\infty)$ *with mean 0 and standard deviation 2. For the purchasing price movement, we let* $\mu = .15$, $\sigma = .5$, $T = 20$ weeks.

We run the simulation 50 times and table 6 provides the averages for the purchasing time, purchasing cost order quantity, selling price, and expected profit of "purchasing at the highest expected profit" method is on the left side, and the averages for the purchasing time, purchasing cost, order quantity, selling price, and expected profit of "purchasing at the lowest expected cost" method is on the right side. Through the table, we notice the expected profit of "purchasing at the highest expected profit" is higher than "purchasing at the lowest cost" when holding cost is .035 and higher. When the holding cost is .04, the expected profit of "purchasing at the highest expected profit" is about 11% higher than the expected profit of "purchasing at the lowest expected cost". So "purchasing at the lowest expected cost" may lead to much lower expected profit than the "purchasing at the highest expected profit" method.

Table 6 also provides averages for the purchasing time, order quantity, selling price, and expected profit during time period $[0,100]$ using our strategy when the holding cost percentage changes from .01 to .04. From intuitive, the expected profit should be always increasing as holding cost percentage decreases. From the table, however, when h is .35, the expected profit jump up. There may be several reasons behind that. First, we define h as the purchasing cost percentage, even if h increases, if the purchasing cost is lower, the expected profit will still increase. Second, our method is only a close-form solution instead of the optimal solution, the results only reflect the outcomes of our method. Third, our sample size is 50, but the potential paths are countless, so the results in table 6 can only describe the rough numbers of expected profit.

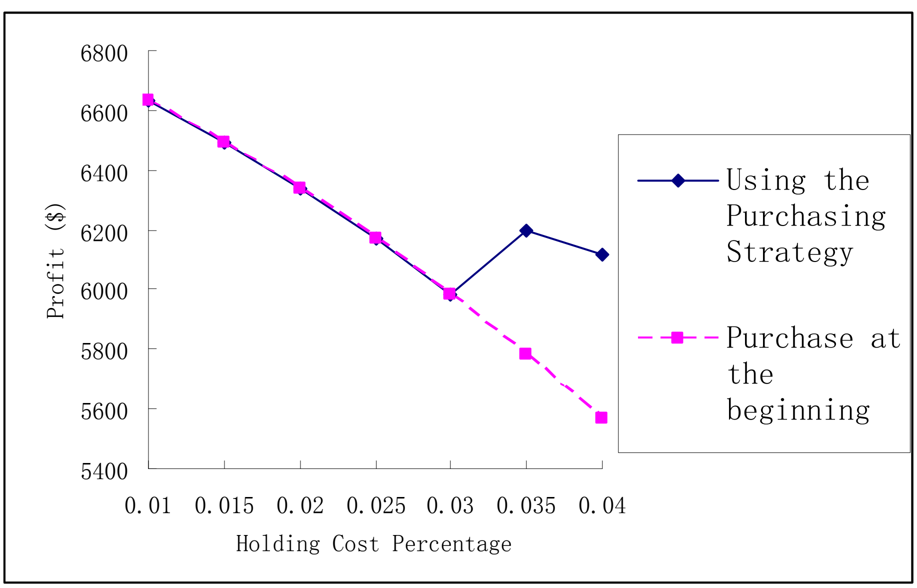
From the table, when holding cost percentage is .035 and .04, the ideal purchasing strategy at time 0 is to wait for some time instead of purchasing right away. When h is less than .3, since it is not so costly to hold the inventory, it is better to purchase right away and the corresponding expected profit increases pretty smoothly to cover more holding cost.

Table 6. Profit Comparison for Changing Holding Cost

Holding Cost Percent.	Maximize Expected Profit					Minimize Expected Cost				
	Avg. Profit	Avg. Quant	Avg Price	Avg Time	Avg Cost	Avg Profit	Avg Quant.	Avg Price	Avg Time	Avg Cost
0.01	6633.2	183.8	43.79	1	7.3891	6633.2	183.8	43.79	1	7.3891
0.015	6491.3	181.76	44.179	1	8.1662	6491.3	181.76	44.179	1	8.1662
0.02	6336.4	179.51	44.608	1	9.025	6336.4	179.51	44.608	1	9.025
0.025	6167.4	177.03	45.083	1	9.9742	6167.4	177.03	45.083	1	9.9742
0.03	5983.3	174.3	45.608	1	11.023	5983.3	174.3	45.608	1	11.023
0.035	6195.6	176.8	45.151	4.04	18.922	5783.2	171.3	46.187	1	12.182
0.04	6116.8	175.89	45.315	4.24	20.109	5566.1	167.99	46.828	1	13.464

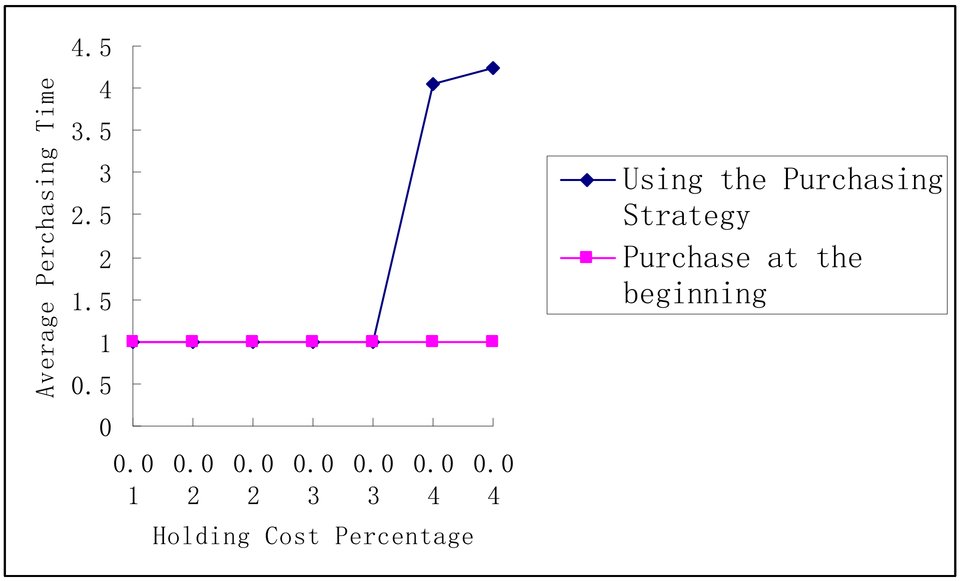
Figure 14 further illustrates the corresponding profit increase trend when holding cost percentage increased from .01 to .04, we observe that the overall trend of expected profit is decreasing but there is a small jump up when holding cost percentage is .035. The result shows the differences between practice and theorem, quick solution process and optimal solution, and numerical example and theoretical results.

Figure 14. Expected Profit Trend when Holding Cost Percentage Increases



From Figure 15, when holding cost percentage is .035 and .04, the ideal purchasing strategy at time 0 is to wait for some time instead of purchasing right away. When h is less than .3, since it is not so costly to hold the inventory, it is better to purchase right away and the corresponding expected profit increases pretty smoothly to cover more holding cost.

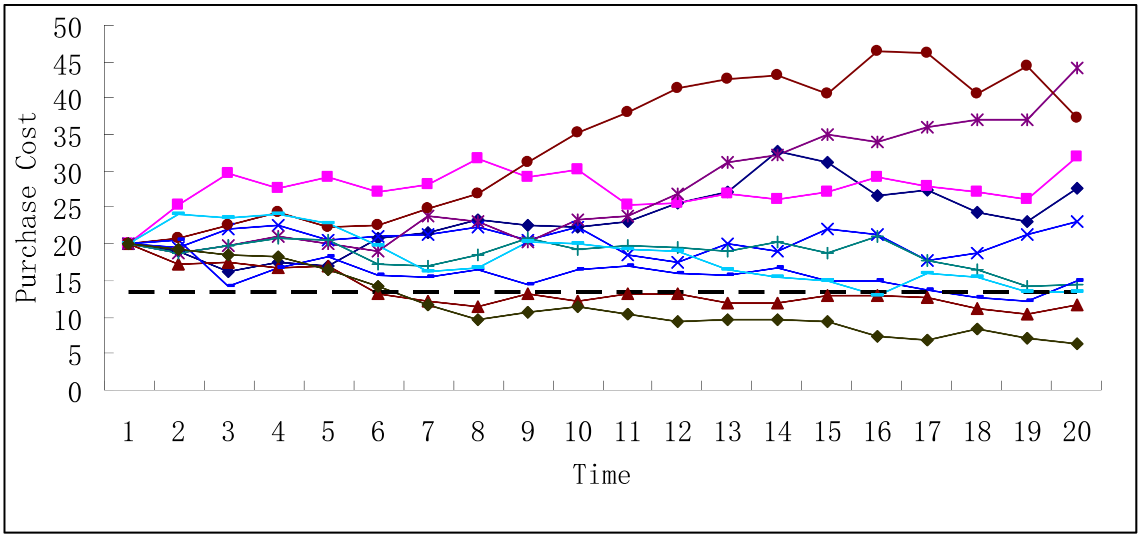
Figure 15. Purchasing Time Trend when Holding Cost Percentage Increases



CASE IV: $Demand = 300 - 5 selling\ price + \varepsilon$, *$\varepsilon$ is normal with mean 0 and standard deviation 2. For the purchasing price movement, let* $\mu = 0$, $\sigma = .1$, $h = .005$ *,and T = 20* weeks.

In this case, we run simulate 10 times and try to have a view at multiple purchasing strategies. First, we draw the plot of these ten paths of the unit cost before selling season (purchasing price and holding cost together) as Figure 16. The trend is zero with diverse variance. Here we assume that we need at least 100 units for the production and have purchase these 100 units at the beginning. We plan to make a second purchase if the unit cost drops from the present $20 to below $13.5 before the selling season. The dashed line shows the target purchasing cost line.

Figure 16. Numerical Examples of Unit Cost (Before Selling Season) Paths



We have already purchased 50 units at $20 unit cost, in the second purchasing, let us purchase additional 25, 50, 75, 100, 125, 150, 175, 200, 225, 250, and 300 units when the cost goes below $13.5 and see how the second purchasing impacts the profit. From the plot, we notice that the target cost have been reached four times in these ten times (Theoretically the probability is .371) and the corresponding costs for these four times are $13.12, $12.64, $12.95, and $11.67. The corresponding average unit cost (average cost for both purchasing), optimal selling price, and the optimal profits for these four purchasing are as following:

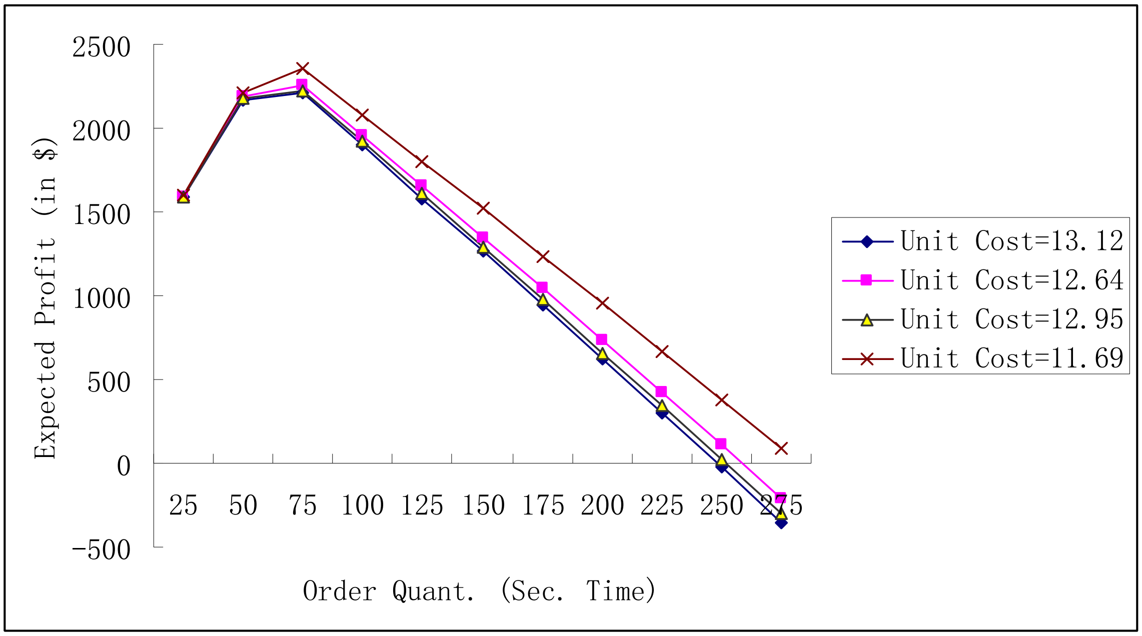
Table 7. Numerical Example of the Second Purchasing Decision

	Second Purchasing. Unit Cost=13.12			Second Purchasing. Unit Cost=12.95			Second Purchasing. Unit Cost=12.64			Second Purchasing. Unit Cost=11.67		
Addit. Order Quant.	Avg. Unit Cost	Sell. Price	Exp. Profit	Avg. Unit Cost	Sell. Price	Exp. Profit	Avg. Unit Cost	Sell. Price	Exp. Profit	Avg. Unit Cost	Sell. Price	Exp. Profit
25	17.7	38.9	1586	17.7	38.8	1588.1	17.5	38.8	1592	17.2	38.6	1604.1
50	16.6	38.3	2172	16.5	38.2	2176.2	16.3	38.2	2184	15.8	37.9	2208.2
75	15.9	37.8	2209	15.8	37.8	2225.7	15.6	37.7	2256.2	15	37.4	2350.9
100	15.4	37.6	1898.7	15.3	37.6	1920	15.1	37.4	1958.7	14.4	37.1	2079.3
125	15.1	37.4	1583	15	37.4	1608.8	14.7	37.3	1655.6	14.1	36.9	1801.5
150	14.8	37.3	1264.1	14.7	37.3	1294.2	14.5	37.1	1349.1	13.8	36.8	1519.9
175	14.6	37.2	943.04	14.5	37.2	977.54	14.3	37	1040.3	13.5	36.7	1235.9
200	14.5	37.1	620.53	14.4	37.1	659.37	14.1	37	730.07	13.3	36.6	950.31
225	14.4	37.1	296.98	14.2	37	340.14	14	36.9	418.7	13.2	36.5	663.5
250	14.3	37	-27.34	14.1	37	20.118	13.9	36.8	106.53	13.1	36.4	375.83
275	14.2	37	-352.2	14	36.9	-300.5	13.8	36.8	-206.3	13	36.4	87.498

From table 7, we notice that the optimal selling price drops as we order more, and the lower unit cost we spend in the second purchasing, the less the selling price and the more expected profit. For example, as the unit cost decreases from 13.12 to 12.95, then to 12.64, and then to 11.57, the optimal selling prices for order 150 units in the second purchasing are 37.3, 37.3, 37.1, and 36.8 correspondingly and the corresponding expected profit are 1264.1, 1294.2, 1349.1 and 1519.9. The same rule applies to the other order quantity. Figure 17 further shows how the order quantity in the second purchasing influences our expected profits. We notice that in these four cases, ordering additional 75 units brings the highest expected profit. The corresponding expected profits are then 2209, 2225.7, 2256.2, and 2350.9.

If we only purchase 50 units at the very beginning without a second purchasing, the optimal selling price is 40 and the expected profit is about 1000. If we purchase an additional 75 unit if the unit cost drops below 12.6, the average expected profit for these ten trials are (2209+2225.7+2256.2+2350.9+1000*6)/10=$1504.18.

Figure 17. Numerical Examples of Expected Profits of Second Purchasing



CASE V: *Demand* $= g - b * \textit{selling price} + \varepsilon$, ε *is uniform distribution with density function* $f(\varepsilon) = \begin{cases} \frac{1}{A} & 0 < \varepsilon < A \\ 0 & \textit{otherwise} \end{cases}$., $T = 30$ weeks, *and* $h = .007\,\textit{purchase price}/\textit{week}$. $\sigma = .1$

In this case, we ran the simulation 50 times and calculated the averages of the simulation results.

We also let the dispose cost v and shortage cost s to be zero. Then equation (9) can be written as $z^*(p \mid c) = A(1 - \frac{c}{p})$ and (10) can be simplified as $g - 2bp + \frac{A}{2} - \frac{Ac^2}{p^2} = 0$. We can get the optimal selling price $p^*(c)$ by solving this equation, and then $q^* = z^*(p^* \mid c) + g - bp$. The corresponding profit can be simplified as

$$E[\Pi(z^*, p^* \mid c)] = (p^* - c)(g - bp^*) + .5Ap(1 - \frac{c}{p})^2 .$$

From the analysis in the previous section, fixing some parameters of selling season will not influent the properties of the analysis and can be easily extended to other cases. In the following numerical example, we specifically define b=.004, g=6.667, A=10, k=5000, and present expected profit is 1,253.651.

In this example, we try a different distribution and let σ be a small one. We find out the outputs are consistent with our theories and are consistent with the observations in previous numerical examples. Table 8 lists the results of comparing "purchasing at the highest expected profit" with "purchasing at the lowest expected cost". The averages for the purchasing time, order quantity, selling price, and expected profit of "purchasing at the highest expected profit" method which are on the left side of the table are different from the averages for the purchasing time, order quantity, selling price, and expected profit of "purchasing at the lowest expected cost" method which are on the right side of the table. In the table, the expected profit of "purchasing at the highest expected profit" is a little higher than "purchasing at the lowest cost" when purchasing cost trend μ is between .07 and .22. When purchasing cost trend μ is lower than 0 or higher than .15 , these two results are the same. The expected profit of "purchasing at the highest expected profit" is not more than 1.1% higher than the expected profit of "purchasing at the lowest expected cost" in this example. So "purchasing at the lowest expected cost" may lead to lower expected profit than the "purchasing at the highest expected profit" method, but the differences are not too much. It satisfies our estimation that these two methods are quite consistent when σ is small. When $\theta > -\sigma / 2$, i.e., the cost trend μ is more than holding cost parameter h ($h = .07$) in this case, our purchasing depends on how much is the difference of μ and h. If μ is much higher than h, we'd better purchase at the very beginning, otherwise, wait a little bit later to make a purchase. If $\theta < -\sigma/2$, i.e., the purchasing cost is increasing very slowly and is lower than the holding cost, the two methods come to the same result and that is "purchasing at the end".

Table 8 also provides averages for the purchasing time, order quantity, selling price, and expected profit during time period $[0,30]$ using our strategy when the cost trend changes from -.03 to .67. From the table, when μ is no more than .22, the higher μ is, the lower will be the expected profit and order quantity, and the higher the potential selling price will be. The ideal purchasing strategy at time 0 is to wait for some time instead of purchasing right away. When μ is more than .22, it is better to purchase right away and the corresponding expected profit is $1253.251. So for a slightly positive cost trend, even if the cost is increasing, it is still possible to gain better profit by waiting for some time to catch the chance to purchase at a lower cost from the fluctuating market. As μ increases, the profit increases, but the purchasing time increases at a decreasing rate, *i.e.,* we can wait for a short time but we'd better not purchase too late.

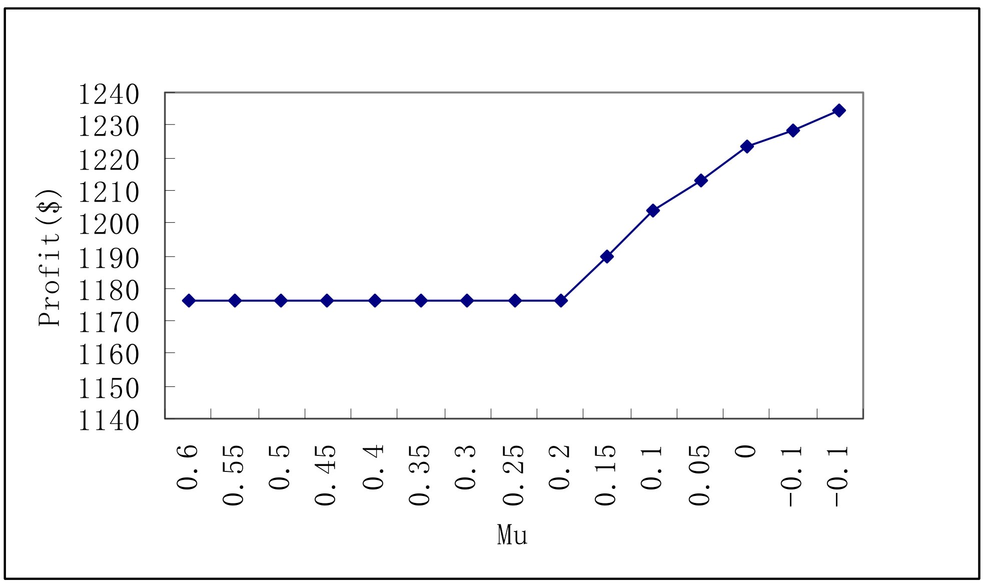
Table 8. Profit Comparison for Changing Drift Term (Uniform Distribution)

μ	**Maximize Expected Profit**				**Minimize Expected Cost**			
	Avg. profit	Avg. Quant.	Avg. Price	Avg. Time	Avg. profit	Avg. Quant.	Avg. Price	Avg. Time
0.67	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.62	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.57	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.52	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.47	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.42	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.37	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.32	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0
0.27	1176.382	9.969	1253.651	0	1176.382	9.969	1253.651	0

0.22	1189.56	9.974	1253.001	5.17	1176.382	9.969	1253.651	0
0.17	1203.532	9.979	1252.311	11.9	1176.382	9.969	1253.651	0
0.12	1212.88	9.983	1251.847	17.22	1176.382	9.969	1253.651	0
0.07	1223.452	9.987	1251.323	28.75	1176.382	9.969	1253.651	0
0.02	1230.221	9.991	1251.174	30	1230.221	9.991	1251.174	30
-0.03	1234.24	9.994	1250.245	30	1234.24	9.994	1250.245	30

Figure 18 further illustrates the corresponding profit increase trend when μ decreases from .22 to -.03, we observe that as μ decreases, the profit increase in a decreasing rate. Sometime it decreases fast and sometimes it decreases slower. When μ is higher than .22, the purchasing strategy is purchasing at the beginning and the expected profits are the present expected profit 1253.651.

Figure 18. Numerical Example of Expected Profit Trend when Cost Trend Decreases



5.2. Profit Impacts of Adding Potential Suppliers

We let $T = 100$, $y(p) = 700 - .2p$, $\varepsilon \sim N(0,1)$, $k = 200{,}000$, and the unit holding cost percentage $h = .001$. We further assumed there were 9 suppliers. For each supplier, $\sigma_1 = \sigma_2 = ... = \sigma_9 = 0.1$, and $\mu_1 = \mu_2 = ... = \mu_9 = 0.6$. At time 0 the purchase costs for supplier 1,2,3…,9 are exp(6), exp(6.1), exp(6), exp(5.8), exp(6.05), exp(6.05), exp(6.05), exp(6.05), exp(6.05) correspondingly. We ran the simulation 25 times and calculated the averages from the simulation results. Table 9 provides simulation results for the purchasing time, order quantity, selling price, and profit during time period $[0,100]$ using our strategy when the potential suppliers increases from only1 supplier to all nine. From the simulation, we found that when the number of suppliers increased from 1 to 9, the average purchase price drops from $403.43 to $109.29, the order quantity increases from 310.7 to 340.81, the average selling price drops from $1,952.8 to $1,805.71, the corresponding average profits increase from $279,816.8 to$375,608.8, and the ideal purchasing time moves from the very beginning to close to the end. This indicates that when there are more potential suppliers, it is possible to wait to catch the chance to purchase at a lower cost from any of the suppliers, and then the company can order more and gain a higher expected profit from a larger demand by charging a lower selling price.

Table 9. Profit Comparison from the Numerical Experiment for the Increasing Supplier Base

No of Suppliers	Fixed Cost	Purchasing price	Purchasing Time	Order Quantity	Selling Price	Profit
1	200,000	403.43	0	310.7	1,952.8	279,816.8
2	200,000	243.66	58.81	326.96	1,872.92	331,050.4
3	200,000	193.97	79.48	332.06	1,848.08	347,542.9
4	200,000	162.77	91.15	335.29	1,832.46	357,973
5	200,000	146.22	91.81	336.43	1,824.18	363,239.7
6	200,000	130.22	92.15	338.64	1,816.18	368,651.4
7	200,000	122.93	92.78	339.41	1,812.53	371,106.5
8	200,000	111.9	94.11	340.54	1,807.02	374,751.4
9	200,000	109.29	94.93	340.81	1,805.71	375,608.8

Figures 19 and 20 further illustrate the increase in both profit and purchasing time using our method when the number of suppliers increased from 1 to 9. From Figures 19 and 20, we observe that as the number of suppliers increases, the profit increases but at a decreasing rate, and the purchasing time also increases at a decreasing rate.

Figure 19. Numerical Example of Profit Trend when Increasing the Supplier Base

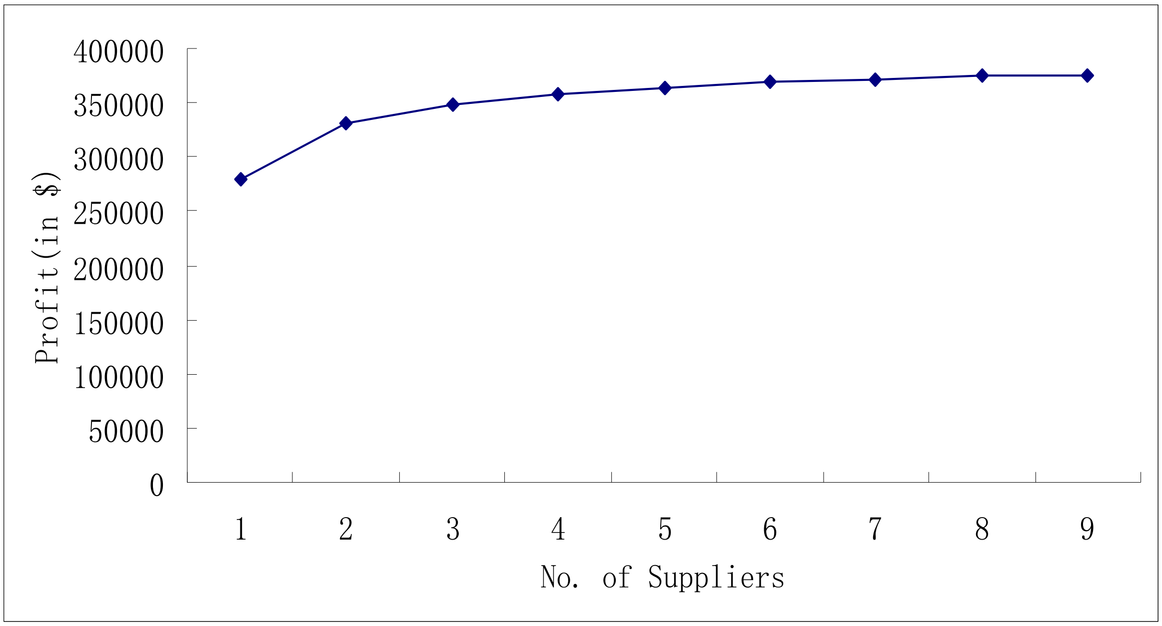
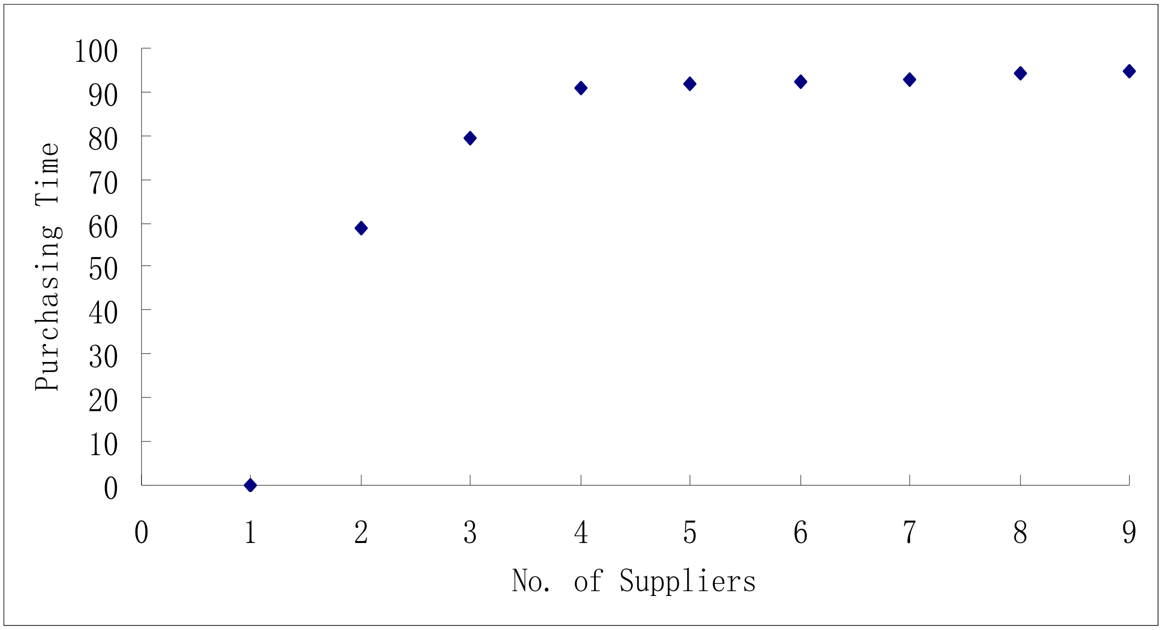


Figure 20. Numerical Example of Purchasing Time Trend When Increasing the Supplier Base



5.3. Risk Impact of Adding Potential Suppliers and Increasing Cost Trends

let T = 100 days and the unit holding cost percentage $h = .001 * purchase\ price / day$. We further assumed that there were 9 suppliers. For the cost trend of each supplier, $\sigma_1 = \sigma_2 = ... = \sigma_9 = 0.1$, and $\mu_1 = \mu_2 = ... = \mu_9 = \tilde{\mu}$, where $\tilde{\mu}$ varied according to the key in Figure 21 to 23. At time 0 the purchase costs for supplier 1,2,3…,9 are exp(1), exp(1.1), exp(.8), exp(.9), exp(1.05), exp(1.05), exp(1.05), exp(1.05), exp(1.05) correspondingly. Here we try to compare how the coefficient in the price sensitive demand function and the target profit influence the downside risks.

Case I: $demand = 3.7 - .1 * selling\ \ price + \varepsilon$, where ε is a truncate normal in $(0, +\infty)$ with mean 0 and variance 1. The target profit is \$42.

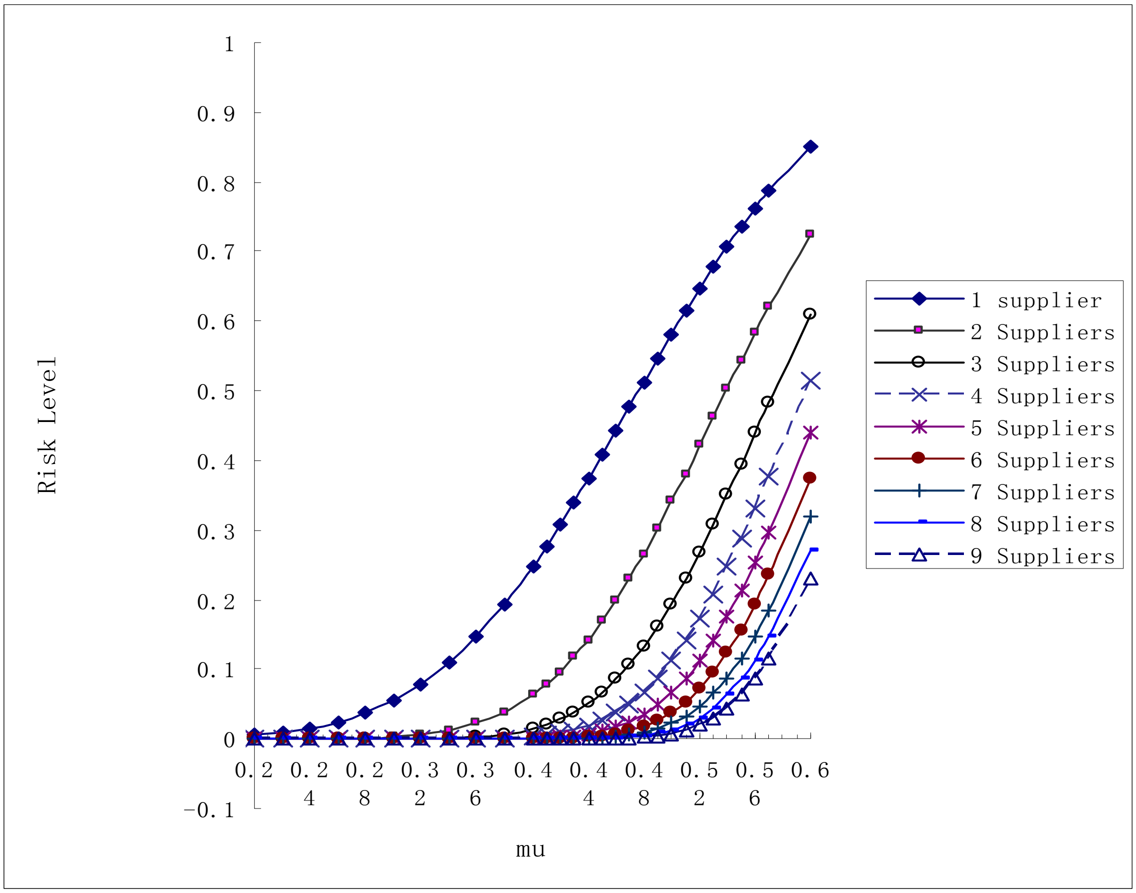
The optimal purchasing strategy was to purchase right away when the price of any supplier caused the profit to drop below the target profit. We ran the simulation 100 times and calculated the averages from the simulation results. Figure 21 illustrates how the corresponding risks, the probabilities of not reaching the target profit $42 as the cost trend α increases from .2 to .6 by .02, drop as the number of supplier increases.

Table 10. Numerical Example I of Downside Risk when Increasing the Supplier Base

	Number of Suppliers								
α	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.2	0.0055	3E-05	2E-07	0	0	0	0	0	0
0.22	0.0093	9E-05	8E-07	0	0	0	0	0	0
0.24	0.0152	0.0002	3E-06	1E-07	0	0	0	0	0
0.26	0.0241	0.0006	1E-05	3E-07	0	0	0	0	0
0.28	0.0368	0.0014	5E-05	2E-06	1E-07	0	0	0	0
0.3	0.0546	0.0031	0.0002	8E-06	5E-07	0	0	0	0
0.32	0.0784	0.0063	0.0005	4E-05	3E-06	2E-07	0	0	0
0.34	0.1093	0.0122	0.0013	0.0001	1E-05	2E-06	2E-07	0	0
0.36	0.1477	0.0223	0.0032	0.0005	7E-05	1E-05	1E-06	2E-07	0
0.38	0.194	0.0384	0.0072	0.0014	0.0003	5E-05	1E-05	2E-06	4E-07
0.4	0.2479	0.0625	0.015	0.0036	0.0009	0.0002	6E-05	1E-05	4E-06
0.41	0.2773	0.0782	0.021	0.0057	0.0016	0.0004	0.0001	4E-05	1E-05
0.42	0.3083	0.0966	0.0288	0.0087	0.0027	0.0008	0.0003	8E-05	3E-05
0.43	0.3406	0.1177	0.0389	0.013	0.0045	0.0015	0.0005	0.0002	6E-05
0.44	0.3739	0.1418	0.0515	0.019	0.0071	0.0027	0.001	0.0004	0.0001
0.45	0.408	0.1687	0.067	0.027	0.0111	0.0045	0.0019	0.0008	0.0003
0.46	0.4427	0.1985	0.0856	0.0374	0.0167	0.0074	0.0033	0.0015	0.0007
0.47	0.4776	0.2309	0.1076	0.0508	0.0244	0.0117	0.0056	0.0027	0.0013
0.48	0.5125	0.2657	0.1331	0.0674	0.0348	0.0179	0.0092	0.0048	0.0025
0.49	0.5472	0.3026	0.162	0.0877	0.0482	0.0265	0.0146	0.008	0.0044
0.5	0.5812	0.3412	0.1943	0.1118	0.0653	0.0381	0.0223	0.013	0.0076
0.51	0.6145	0.3811	0.2298	0.1399	0.0863	0.0533	0.0329	0.0203	0.0125
0.52	0.6467	0.4218	0.268	0.1718	0.1116	0.0725	0.0471	0.0306	0.0199
0.53	0.6776	0.4629	0.3085	0.2074	0.1411	0.096	0.0653	0.0444	0.0302
0.54	0.7072	0.5038	0.3509	0.2463	0.1748	0.1241	0.0881	0.0625	0.0444

0.55	0.7352	0.5442	0.3945	0.288	0.2125	0.1567	0.1156	0.0853	0.0629
0.56	0.7616	0.5837	0.4387	0.3319	0.2536	0.1937	0.148	0.1131	0.0864
0.57	0.7862	0.6217	0.4829	0.3774	0.2976	0.2346	0.185	0.1459	0.115
0.6	0.8495	0.7249	0.6101	0.5159	0.4392	0.374	0.3184	0.2711	0.2308

Figure 21. Risk Trends when Increasing the Supplier Bases (Example I)



From the table and the figure, we can see that the risk decreases at a decreasing rate as the number of suppliers increases. As the purchasing cost trend μ increases, the risk increases. As there are 1, 2, 3…9 suppliers, the risks for one supplier is around 0 when purchasing cost trend $\mu = .2$, when purchasing cost trend increase to $\mu = .6$, the downside risk for 1, 2, 3, 4, 5, 6, 7, 8, and 9 are about .85, .72, .61, .52, .44, .37, .32, .27, and .23

respectively. It shows clearly that the downside risk increases quickly when the purchasing cost trend is more upward and increasing supplier bases will effectively decrease our downside risk. We can easily decide the necessary number of suppliers for a certain risk level. For instance, in this example, choosing three suppliers decreased the risk to less than 20% when $\mu = .5$.

Case II: $demand = 3.7 - .01 * selling\ price + \varepsilon$ *, where* ε *is a truncate normal in* $(0, +\infty)$ *with mean 0 and variance 1. The target profit is $420.*

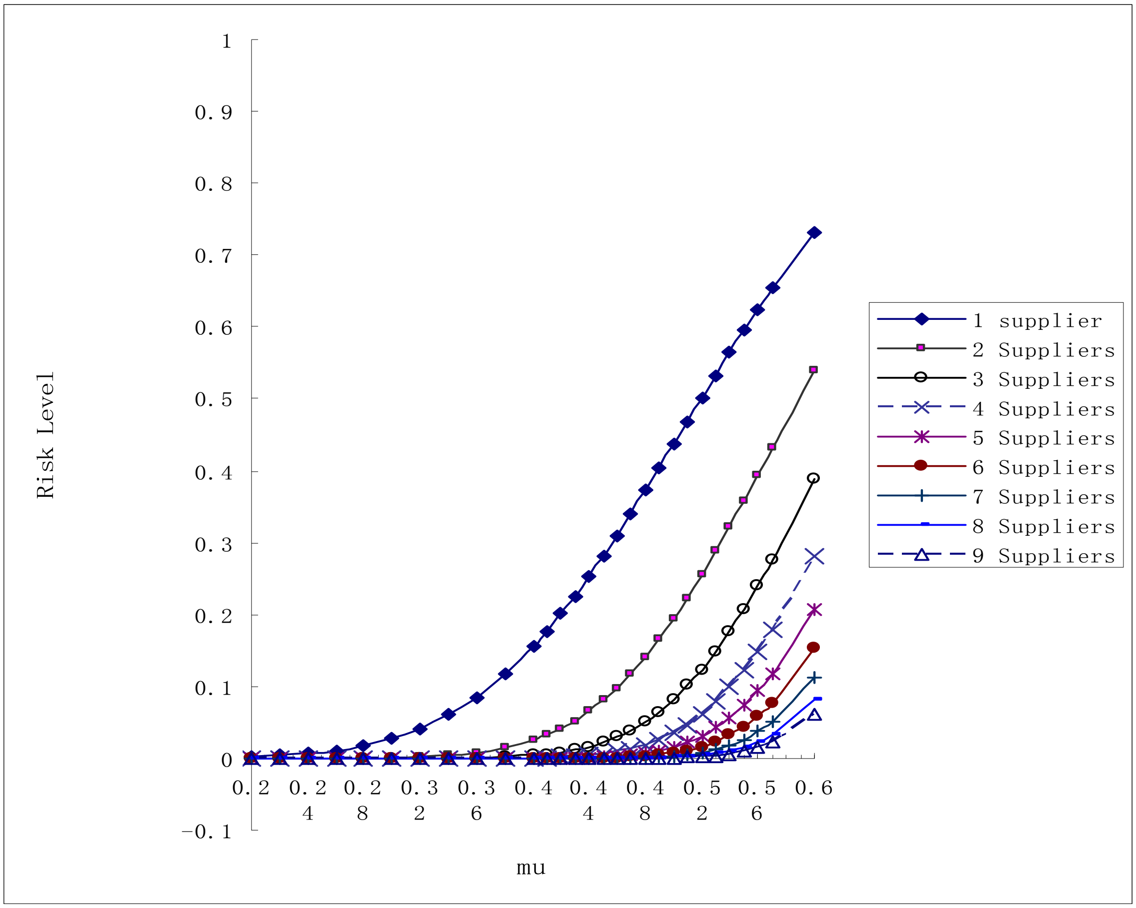
The optimal purchasing strategy was to purchase right away when the price of any supplier caused the profit to drop below the target profit. We ran the simulation 100 times and calculated the averages from the simulation results. Figure 22 illustrates how the corresponding risks, the probabilities of not reaching the target profit $420 as the cost trend α increases from .2 to .6 by .02, drop as the number of supplier increases.

From the table and the figure, we can see that the risk decreases at a decreasing rate as the number of suppliers increases. As the purchasing cost trend μ increases, the risk increases. As there are 1, 2, 3…9 suppliers, the risks for one supplier is are all around 0 when purchasing cost trend $\mu = .2$, when purchasing cost trend increase to $\mu = .6$, the downside risk for 1, 2, 3, 4, 5, 6, 7, 8, and 9 are about .73, .54, .38, .28, .20, .15, .11, .08, and .06 respectively. It shows clearly that the downside risk increases quickly when the purchasing cost trend is more upward and increasing supplier bases will effectively decrease our downside risk. We can easily decide the necessary number of suppliers for a certain risk level. For instance, in this example, choosing three suppliers decreased the risk to less than 8% when $\mu = .5$. Comparing case I, in which the coefficient of the selling price is .1, the selling price has less negative impact to the market demand and the risk drops from 20% to 8%.

Table 11. Numerical Example II of Downside Risk when Increasing the Supplier Base

	Number of Suppliers								
α	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.2	0.0023	5E-06	0	0	0	0	0	0	0
0.22	0.004	2E-05	1E-07	0	0	0	0	0	0
0.24	0.0068	5E-05	3E-07	0	0	0	0	0	0
0.26	0.0113	0.0001	1E-06	0	0	0	0	0	0
0.28	0.018	0.0003	6E-06	1E-07	0	0	0	0	0
0.3	0.0279	0.0008	2E-05	6E-07	0	0	0	0	0
0.32	0.0418	0.0018	7E-05	3E-06	1E-07	0	0	0	0
0.34	0.0606	0.0038	0.0002	1E-05	8E-07	0	0	0	0
0.36	0.0854	0.0075	0.0006	5E-05	4E-06	4E-07	0	0	0
0.38	0.1168	0.014	0.0016	0.0002	2E-05	3E-06	3E-07	0	0
0.4	0.1553	0.0247	0.0037	0.0006	9E-05	1E-05	2E-06	3E-07	1E-07
0.41	0.1771	0.0321	0.0054	0.0009	0.0002	3E-05	5E-06	1E-06	2E-07
0.42	0.2007	0.0412	0.0079	0.0016	0.0003	6E-05	1E-05	3E-06	5E-07
0.43	0.2259	0.0521	0.0113	0.0025	0.0006	0.0001	3E-05	7E-06	1E-06
0.44	0.2527	0.0652	0.0158	0.0039	0.001	0.0003	7E-05	2E-05	4E-06
0.45	0.2809	0.0804	0.0217	0.006	0.0017	0.0005	0.0001	4E-05	1E-05
0.46	0.3103	0.0981	0.0293	0.0089	0.0028	0.0009	0.0003	9E-05	3E-05
0.47	0.3407	0.1182	0.0388	0.013	0.0045	0.0015	0.0005	0.0002	6E-05
0.48	0.372	0.1407	0.0506	0.0185	0.0069	0.0026	0.001	0.0004	0.0001
0.49	0.4039	0.1658	0.0648	0.0257	0.0105	0.0043	0.0017	0.0007	0.0003
0.5	0.4361	0.1932	0.0816	0.035	0.0154	0.0068	0.003	0.0013	0.0006
0.51	0.4686	0.2228	0.1013	0.0467	0.0221	0.0104	0.0049	0.0023	0.0011
0.52	0.5009	0.2544	0.1238	0.0611	0.0308	0.0156	0.0078	0.004	0.002
0.53	0.5329	0.2877	0.1492	0.0784	0.0421	0.0226	0.0121	0.0065	0.0035
0.54	0.5643	0.3225	0.1773	0.0988	0.0561	0.0319	0.0181	0.0103	0.0058
0.55	0.595	0.3583	0.208	0.1223	0.0732	0.0438	0.0262	0.0157	0.0094
0.56	0.6248	0.3948	0.241	0.1489	0.0935	0.0588	0.0369	0.0232	0.0146
0.57	0.6536	0.4317	0.276	0.1784	0.1172	0.077	0.0506	0.0333	0.0218
0.6	0.7321	0.5407	0.3886	0.282	0.2073	0.1525	0.1121	0.0825	0.0606

Figure 22. Risk Trends when Increasing the Supplier Bases (Example II)



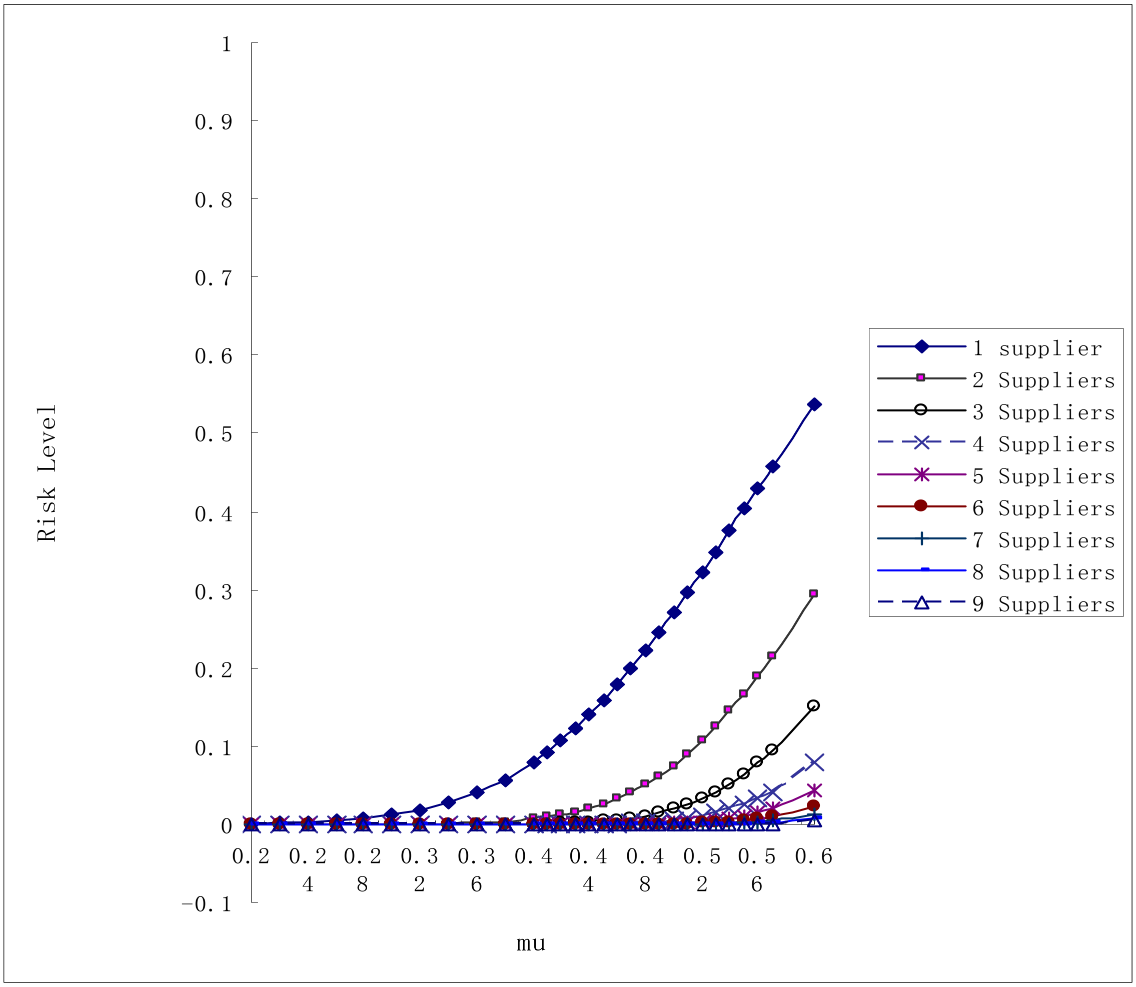
Case III: demand $= 3.7 - .001 * selling\ price + \varepsilon$ *, where* ε *is a truncate normal in* $(0, +\infty)$ *with mean 0 and variance 1. The target profit is $4200.*

The optimal purchasing strategy was to purchase right away when the price of any supplier caused the profit to drop below the target profit. We ran the simulation 100 times and calculated the averages from the simulation results. Figure 23 illustrates how the corresponding risks, the probabilities of not reaching the target profit $4200 as the cost trend α increases from .2 to .6 by .02, drop as the number of supplier increases.

Table 12. Numerical Example III of Downside Risk when Increasing the Supplier Base

	Number of Suppliers								
α	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.2	0.0007	6E-07	0	0	0	0	0	0	0
0.22	0.0014	2E-06	0	0	0	0	0	0	0
0.24	0.0025	6E-06	0	0	0	0	0	0	0
0.26	0.0042	2E-05	1E-07	0	0	0	0	0	0
0.28	0.0071	5E-05	3E-07	0	0	0	0	0	0
0.3	0.0114	0.0001	1E-06	0	0	0	0	0	0
0.32	0.0179	0.0003	5E-06	1E-07	0	0	0	0	0
0.34	0.0272	0.0008	2E-05	5E-07	0	0	0	0	0
0.36	0.04	0.0017	6E-05	2E-06	1E-07	0	0	0	0
0.38	0.057	0.0034	0.0002	1E-05	6E-07	0	0	0	0
0.4	0.079	0.0065	0.0005	4E-05	3E-06	2E-07	0	0	0
0.41	0.092	0.0088	0.0007	7E-05	6E-06	6E-07	1E-07	0	0
0.42	0.1064	0.0117	0.0012	0.0001	1E-05	1E-06	2E-07	0	0
0.43	0.1223	0.0155	0.0018	0.0002	3E-05	3E-06	4E-07	0	0
0.44	0.1395	0.0201	0.0026	0.0004	5E-05	7E-06	1E-06	1E-07	0
0.45	0.1582	0.0258	0.0038	0.0006	9E-05	2E-05	2E-06	4E-07	1E-07
0.46	0.1782	0.0328	0.0055	0.0009	0.0002	3E-05	6E-06	1E-06	2E-07
0.47	0.1996	0.041	0.0077	0.0015	0.0003	6E-05	1E-05	3E-06	5E-07
0.48	0.2221	0.0508	0.0106	0.0023	0.0005	0.0001	3E-05	6E-06	1E-06
0.49	0.2458	0.0622	0.0144	0.0034	0.0009	0.0002	5E-05	1E-05	3E-06
0.5	0.2704	0.0752	0.0192	0.005	0.0014	0.0004	0.0001	3E-05	8E-06
0.51	0.2959	0.0899	0.0252	0.0072	0.0022	0.0007	0.0002	6E-05	2E-05
0.52	0.3221	0.1065	0.0325	0.0102	0.0033	0.0011	0.0004	0.0001	4E-05
0.53	0.3488	0.1248	0.0413	0.014	0.005	0.0018	0.0006	0.0002	8E-05
0.54	0.3759	0.1448	0.0517	0.019	0.0072	0.0027	0.001	0.0004	0.0002
0.55	0.4032	0.1664	0.0639	0.0251	0.0103	0.0042	0.0017	0.0007	0.0003
0.56	0.4305	0.1896	0.0778	0.0327	0.0143	0.0062	0.0027	0.0012	0.0005
0.57	0.4577	0.2141	0.0936	0.0419	0.0194	0.009	0.0042	0.0019	0.0009
0.6	0.5371	0.2942	0.1516	0.0798	0.0433	0.0235	0.0127	0.0069	0.0037

Figure 23. Risk Trends when Increasing the Supplier Bases (Example III)



From the table and the figure, we can see that the risk decreases at a decreasing rate as the number of suppliers increases. As the purchasing cost trend μ increases, the risk increases. As there are 1, 2, 3…9 suppliers, the risks for one supplier is are all around 0 when purchasing cost trend $\mu = .2$, when purchasing cost trend increase to $\mu = .6$, the downside risk for 1, 2, 3, 4, 5, 6, 7, 8, and 9 are .54, .29, .15, .08, .04, .02, .01, .01, and .00 respectively. It shows clearly that the downside risk increases quickly when the purchasing cost trend is more upward and increasing supplier bases will effectively decrease our downside risk. We can easily decide the necessary number of suppliers for a certain risk level. For instance, in this example, choosing three suppliers decreased the risk to less than 8% when $\mu = .5$.

From these three cases, we notice that the downside risk is very sensitive to the impact factor of selling price to demand, *i.e.*, *b* in the demand function $D = y(p) + \varepsilon = g - bp + \varepsilon$. As *b* shrink in ten times from .1 to .01, and then to .001, even the target profit increase in ten times from \$42 for b=.1 to \$420 for b=.01, and then to \$4200 for b=.001, the corresponding downside risks decreases for any μ and number of suppliers. For example, as cost trend α =.2, .3, .4, .5, .6, the downside risks for $y(p) = 3.7 - .1p$ with target profit 42 for purchasing from one supplier are .0007, .0114, .079, .2704, .5371 respectively; the downside risks for $y(p) = 3.7 - .01p$ with target profit 420 for purchasing from one supplier are .0023, .0279, .1553, .43361, .7321 respectively; and the downside risks for $y(p) = 3.7 - .001p$ with target profit 42 for purchasing from one supplier are .0055, .0546, .2479, .5812, .8495 respectively. It seems that as the market demand is more sensitive to the selling price, the downside risk will become progressively larger.

6. Conclusions

In this chapter, I study how to design integrated procurement and selling strategies for a newsvendor to maximize profit, under the assumption that the newsvendor has a specific time period before the commencement of the selling season to make the purchase. The purchasing cost of the raw material fluctuates over time, and the demand for the product is random and price-sensitive. Even though a Binomial Tree approximation is used extensively to solve price uncertainty problems, the solution time of that method increases exponentially with the number of the suppliers and the number of periods. Therefore, we go further and propose another solution algorithm for this specific problem, which substantially decreases its computational complexity.

We extend our solution algorithm to the cases of forward contracts, multiple suppliers, multiple demand points and risk minimization. Whether it is more profitable to choose a forward contract or to purchase directly from the spot market depends on whether the expected profit from the spot market using our strategy exceeds the profit from a forward contract. When facing multiple potential suppliers with potentially different price processes, purchasing from multiple suppliers with potentially different price processes can lead to higher profits due to a higher probability of being able to buy at a low price from at least one of them. When units are stocked to satisfy demands in multiple time points, we discuss the scenarios in which such a problem can be decomposed into several single-period subproblems. We further provide numerical analysis to show how to use Monte Carlo simulation to plan the second purchasing when there is a second purchase option available during the horizon, reveal how and when the profit, purchasing time, and selling price of purchasing at the lowest expected cost differ from those of purchasing at the highest expected profit, reveal that higher unit holding cost rate will postpone the purchase, and demonstrate the profound impact that increasing the potential supplier base can have on profit and risk, together with the effect of these parameters on the purchasing decision.

CHAPTER V: CONCLUSIONS & FUTURE DIRECTIONS

For some products, prices fluctuate constantly and unpredictably. This dissertation provides tools and insights for managers to handle the purchasing decisions for these products effectively. The basic suggested approach is, given a purchasing time horizon, to pick a goal (expected profit or cost), and keep delaying the purchase until that goal has been reached. We apply this idea under scenarios that include identifying the expected optimum time at which the lowest price occurs, planning the procurement with and without quantity flexibility when the selling price is extrinsic, and investigating how to design integrated procurement and selling strategies for a newsvendor.

In chapter II, we derived expressions of the contract's expected low price and its second moment for a given horizon, then we identified an expected optimum time to minimize the expected squared loss. Simulation experiments verified our analysis, and we identified that the expected optimum purchasing time is in the middle of the time horizon when the cost trend is level, at the very beginning when the cost rises sharply, and is at the very end when the cost drops sharply.

In chapter III, we analyzed purchasing strategies for retailers regarding the best timing and amount of purchases when operating under contracts in which the purchasing time and order quantity are flexible and the purchasing price is stochastic. We developed a Time Strategy and a Target Strategy and compared these two strategies in timing flexibility contracts with or without quantity flexibility and then combined these methods into an approximate algorithm to facilitate the purchasing decision in a more efficient way. We then extended the solution procedures to cases of multiple suppliers, minimizing downside-risk, and maximizing profit given a risk level.

Finally, in Chapter IV, we studied the problem of planning the procurement and sales for a newsvendor for whom the price of the raw material fluctuates along time and the demand of the output product is random and price-sensitive. After we provided a backward deduction method to solve this problem, we developed an efficient solution algorithm adapted for multiple-supplier cases and long-term-length scenarios and a corresponding expected lowest profit. We proved that when the cost drops and satisfies certain criteria, the optimal purchasing decision is to purchase at the end. We further revealed that when risk increase dominates the profit increase, a risk-averse (even a risk-neutral) agent will hedge the risk by either purchasing earlier or using option or forward contracts. Furthermore, we discussed the scenarios in which the multiple time points problem can be decomposed into several single-period subproblems and illustrated how purchasing from multiple suppliers with different modes of price movements leads to higher profits.

Much work remains to be done in the study of contracts under uncertain sourcing conditions. For example, further research can extend our analysis to include uncertainties of lead-time and supply capacity. Combining processing flexibility and demand flexibility is also an interesting topic for future research. The research can also be extended to the purchasing, order quantity and selling decisions under multiple objectives of risk control, cost efficiency and profit maximum. Some other possible extensions include studying how to make the decisions of entering and switching among projects facing information update and price uncertainties, examining the make vs. buy decision, designing investment strategies when there are multiple projects available, investigating how to use options to hedge the risk in operations, and making inventory decisions in the EOQ setting in which the purchasing price or the capacity is uncertain.

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APPENDIX: Simulation of Squared Losses in the Stochastic Environment

```
#initial values
n = 100                  #time horizon
h = 2000                 #simulation times
sita = 0.2               #cost trend
u=124                    #starting total cost position minimize E(I-x)^2 for the ith simulation
sigma = 0.1              #standard deviation

X = matrix(0, nrow = h, ncol = n)              # w[i, j]record x value at time j for the
    ith simulation
Difmin = matrix(0, nrow = h, ncol = n)         # Difmin[i, j]=(I(T)-x(j))^2 for the ith
    simulation, that is the squred difference between value at time j and minimum value for
    the ith simulation
I= rep(0,h)              #record the minimum point for the ith simulation
Edif = rep(0,n)          #record the E(I-x(t))^2 for time t
MinT =0                  #record the time that minize E(I-x(t))^2

# Caculate the results for the simulated reality
for(j in 1:h){
    X[j,1]=u             #the process start from value u
```

```
    I[j]=X[j,1]
    for(i in 2:n){
        k=i-1
        X[j,i]=sita+ rnorm(1,mean=0,sd=1) +X[j,k]
        if(X[j,i]<I[j]){   #Find the minimum in the j process
                I[j]=X[j,i]
        }
    }
}
for(j in 1:h){
    for(i in 1:n){
        Difmin[j,i]=(X[j,i]-I[j])^2 #Fine the squred difference between value at time j and
    minimum value for the ith simulation
    }
}

MinVal=mean(Difmin[,1])    #Initial MinVal, MinVal record the minimum E(I-x(t))^2
MinT=1                     #Initial MinT, MinT record the time that minimize E(I-x(t))^2
for(i in 1:n){             #Find the time that minimize E(I-x(t))^2
    Edif[i]=mean(Difmin[,i]) #Find the E(I-x(t))^2 for time t
    if (Edif[i]<MinVal){
        MinT=i                 #MinT record the time that minimize E(I-x(t))^2
        MinVal=Edif[i]         #MinVal record the minimum E(I-x(t))^2
```

```
    }
}

#results
Edif    #the E(I-x(t))^2 for time t
MinT   #the time that minimize E(I-x(t))^2

#plot
tt=1:100                  #time
par(mfrow=c(3,3))
plot(tt,Edif[tt],main=paste("theta=",sita),xlab="time",ylab="E(I-x(t))^2")     #plot
```