Some Observations Toward Decentralized Control Under Saturation

By

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Some Observations Toward Decentralized Control Under Saturation

Abstract

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This dissertation is mainly about discrete-time decentralized control system design. Many large-scale systems' dynamics are characterized by their wide-spread topology and so these systems are considered to be *decentralized*. Subsequently, analysis of the decentralized systems is generally a really complicated task because one should consider all the inter-connections within the structure of such system. Furthermore input saturation is a practical constraint for many existing systems. This document tries to address some of the existing problems regarding analysis and design of decentralized control system. Here we achieve two goals. First we find necessary and sufficient condition for design of linear time invariant decentralized controller for a specific but broad class of linear time invariant system under input saturation. Second, we take one step further and find time-varying decentralized controller for linear time invariant systems.

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To my dear parents Mahasti and Hosein,

1. INTRODUCTION

Actuator saturation is a common phenomenon in many linear centralized and decentralized systems. This common constraint imposes many complications on the study and analysis of these systems. The analysis and design of controllers for centralized linear systems subject to input saturation have been studied for many years [22],[17]. Although there is such a rich literature for centralized LTI control systems, much less effort is given to analysis of linear decentralized systems under input saturation. This thesis explains some parts of the current research on decentralized controller design that has been done here at Washington State University that ar leading toward a comprehensive understanding of decentralized control under saturation. Specifically in chapter 2 an alternative approach to designing stabilizing compensators for saturating linear time-invariant plants is given; This novel compensator architecture perhaps is really beneficial for decentralized-systems stabilization. Necessary and sufficient conditions for the existence of LTI controllers for a specific but broad class of LTI discrete-time decentralized systems under input saturation are sought in chapter 3. Design of decentralized linear time varying controllers for linear time-invariant discrete-time systems is pursued in Chpater 4. Finally conclusions and future directions for research are given in Chapter 5. Now, let us give a few specifications regarding each direction, pursued in the thesis.

First, in chapter 2 a novel method of designing stabilizing compensators for LTI plants under input saturation is introduced. the motivation for this design is that the results can be fruitful to the ongoing studies of decentralized controller design under saturation. Later in this chapter, we merge the new design with time scale notions to construct lower-order controllers for some plants.

Decentralized control of discrete-time linear time-invariant systems with input saturation is studied in Chpter 3. Here we show the necessary conditions for the existence of LTI decentralized controllers for such systems are that all the so called *decentralized fixed modes* of the system are in the open unit disc and all the eigenvalues of the system are in the closed unit disc. We also show that these conditions are sufficient if those eigenvalues of the system that are on the unit disc are not repeated.

Time varying decentralized controllers for discrete-time systems are pursued in chapter 4. Our motivation for doing so is that there exists some linear time-invariant systems for which there do not exist time-invariant, linear decentralized controllers to stabilize them. We show a methodology for the design of linear time-varying decentralized controllers for this class of systems if and only if their so called *quotient fixed modes* are not outside the open unit disc.

2. AN ALTERNATIVE APPROACH TO DESIGNING STABILIZING COMPENSATORS FOR SATURATING LINEAR TIME-INVARIANT PLANTS

We present a new methodology for designing low-gain linear time-invariant (LTI) controllers for semi-global stabilization of an LTI plant with actuator saturation, that is based on representation of a proper LTI feedback using a precompensator plus static-output-feedback architecture [1]. We also mesh the new design methodology with time-scale notions to develop lower-order controllers for some plants.

2.1 Motivation of Problem

Low-gain output feedback stabilization of linear time-invariant plants subject to actuator saturation has been achieved using the classical observer–followed–by–state–feedback controller architecture [2, 3, 4]. In this note, we discuss an alternative controller architecture for designing low-gain output feedback control of linear time-invariant (LTI) plants with saturating actuators. Specifically, we use a classical result of Ding and Pearson to show that a dynamic prefiltering together with static output feedback architecture can naturally yield a stabilizing low-gain controller under actuator saturation (Section 2). Subsequently, by using time-scale notions, we illustrate through a SISO example that lower-order controllers can be designed, in the case where the $j\omega$-axis eigenvalues of the plant are in fact at the origin (Section 3).

The reader may wonder what advantage the alternate architectures provide. Our particular motivation for developing the alternatives stems from our ongoing efforts on decentralized controller design, and in particular our effort to develop a low-gain methodology for decentralized plants [5, 7, 8, 6, 9, 10]. In pursuing this goal, we have needed to use several novel controller structures, in particular ones that utilize pre-compensators and output derivatives together (see our works in [9, 10, 7, 8]). This paper delineates the particular use of the new controller atchitectures in stabilization under saturation.

What this study of decentralized control makes clear is that freedoms in the structure of the controller facilitate design, because they can permit design that fit the structural limitations of the problem (in this case, decentralization). The alternatives to low-gain control that we propose here serve this purpose, because they naturally permit selection of a desirable controller architecture for the task at hand. While our primary motivation is in the decentralized controls arena, we believe that these alternate architectures may also be useful in such domains as adaptive control and plant inversion through lifting [9].

2.2 Low-Gain Output Feedback Control through Precompensation

In this section, we demonstrate design of low-gain proper controllers for semi-global stabilization of LTI plants subject to actuator saturation, using a novel precompensator-based architecture. We also briefly discuss the connection of our design to the traditional observer-based design, and expose that the design is deeply related to a family of precompensator-based designs that also permit e.g. zero cancellation and relocation.

Formally, we demonstrate design of a proper output feedback compensator that achieves semi-global stabilization of the following plant $\mathcal{G}$:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\sigma(\mathbf{u}) \tag{2.1}$$
$$\mathbf{y} = C\mathbf{x},$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $\sigma()$ is the standard saturation function, A has eigenvalues in the closed left-half-plane (CLHP), and the plant is observable and controllable*. Our design is fundamentally based on 1) positing a control architecture comprising a pre-compensator with a zero-free and uniform-rank structure together with a feedback of the output and its derivatives (see Figure 2.1), 2) designing the controller using this architecture, and 3) arguing that the designed controller admits a proper feedback implementation. This controller design directly builds on two early results: 1) Ding and Pearson's result [11] for pole-placement that is based on a dynamic pre-compensation + static feedback representation of a proper controller (Figure 2.1a and 2.1b); and 2) Lin and Saberi's effort [2] on stabilization under saturation using state feedback. For clarity, we cite the two results in the lemmas before we present our main result.

Lemma 1 concerns pre-compensator and feedback design for pole placement in a general LTI system, i.e. one of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{2.2}$$
$$\mathbf{y} = C\mathbf{x},$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$.

* In fact, the methods developed here trivially generalize to the case where the dynamics are stabilizable and detectable. We consider the observable and controllable case for the sake of clarity.

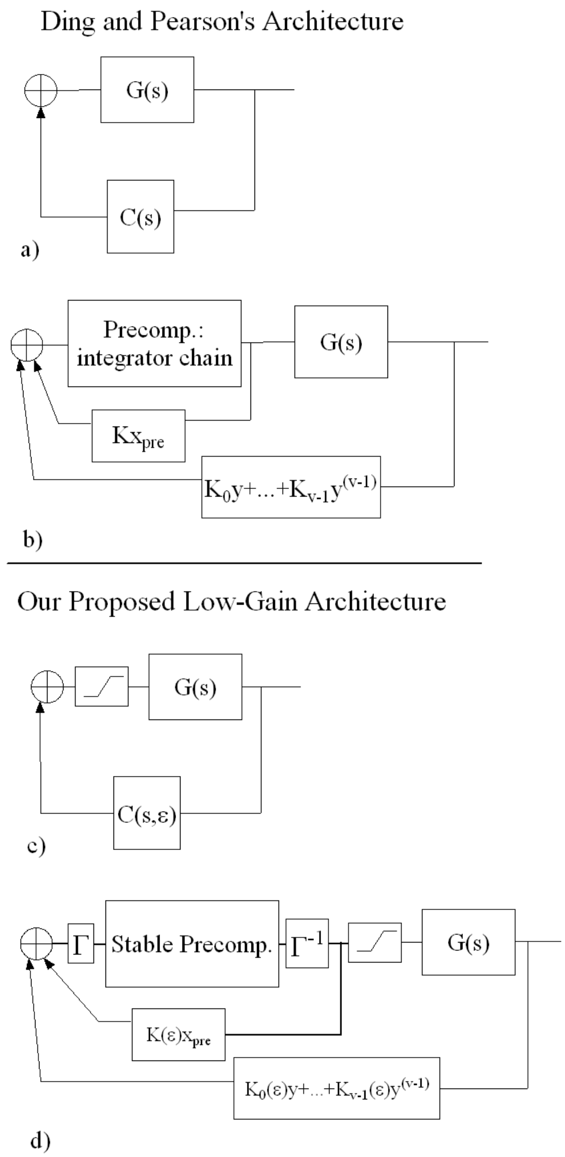

Fig. 2.1: Compensator artechitectures: *a*) and *b*) show the compensator artechitectures presented in Ding and Pearson, in particular, a precompensator-together-with-static feedback viewpoint. *b*) is used to design a proper compensator of *a*). *c*) and *d*) show the compensator artechitectures that stabilize a plant under input saturation.

Lemma 2.1. *Consider a plant of the form (2.2) that is controllable and observable, with observability index v. Pre-compensation through addition of $v-1$ integrators to each plant input permits computation of the plant's state $\mathbf{x}$ as a linear function of the plant's output $\mathbf{y}$, its derivatives up to $\mathbf{y}^{(v)}$ and the pre-compensator's state. A consequence of this computation capability is that it permits design of a proper feedback controller $C(s)$ that places the poles of the compensated plant at arbitrary locations (closed under conjugation).*

When the matrix C in the system (2.2) is not invertible, the classical method to obtain the state information from output is through observer design. This lemma of Ding and Pearson gives an alternative design for state estimation and feedback controller design, that is based on viewing the proper compensator $C(s)$ as a dynamic pre-compensation together with static feedback (Figure 2.1a and 2.1b). Specifically, the methodology of design is as follows: first, from the pre-compensator-based representation (Figure 2.1b), a computation of the plant state from the plant output and its derivatives together with the precompensator state can directly be obtained. Second, the classical state feedback methodology thus permits us to compute the static feedback in the pre-compensator-based representation, so as to place the closed-loop eigenvalues at desired locations. Third, the equivalence between the precompensator-based representation and a proper feedback controller is used to obtain a realization of the feedback control (Figure 2.1a). We kindly ask the reader to see [11, 12], both for the details of the state computation and the equivalence between the precompensator-based architecture and the proper feedback controller. In our development, we broadly replicate the design methodology of Ding and Pearson, but use a stable rather than neutral precompensator in order to obtain a controller that works under input saturation.

Lemma 2 is concerned with using linear state feedback control to semi-globally stabilize the plant:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\sigma(\mathbf{u}), \tag{2.3}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $\sigma()$ is the standard saturation function. Please see Lin and Saberi's work [2] for the proof of the lemma.

Lemma 2.2. *Consider a plant of the form (2.3) that satisfies two conditions: 1) all the eigenvalues of A are located in the CLHP; 2) (A, B) is stabilizable. The plant can be semiglobally stabilized using linear static state feedback. That is, a parametrized family of compensators $u = K(\epsilon)x$ can be designed such that, for any specified ball of plant initial conditions $\mathcal{W}$, there exists $\epsilon^*(\mathcal{W})$ such that, for all $0 < \epsilon \leq \epsilon^*(\mathcal{W})$, the compensator $K(\epsilon)$ achieves local exponential stabilization and contains $\mathcal{W}$ in its domain of attraction.*

Now we are ready to present the main result. Specifically, the following theorem formalizes that a family of proper controllers can be designed for semi-global stabilization of $\mathcal{G}$, based on the precompensator-together-with-derivative-feedback architecture shown in Figure 2.1. The proof of the theorem makes clear the design methodology.

Theorem 2.3. *The plant $\mathcal{G}$ (Equation 1) can be asymptotically semi-globally stabilized using proper feedback compensation of order mv, where v is the observability index of the plant. Specifically, a parametrized family of compensators $C(s, \epsilon)$ can be designed (Figure 2.1c) to achieve the following: for any specified ball of plant and compensator initial conditions $\mathcal{W}$, there exists $\epsilon^*(\mathcal{W})$ such that, for all $0 < \epsilon \leq \epsilon^*(\mathcal{W})$, $C(s, \epsilon)$ is locally exponentially stabilizing and contains $\mathcal{W}$ in its domain of attraction. The design can be achieved by developing a controller of the architecture shown in Figure 2.1d—i.e., comprising an m-input uniform-rank square-invertible zero-free precompensator*

P with input $\mathbf{u}_p$ together with a feedback of the form $\mathbf{u}_p = K_0(\epsilon)\mathbf{y} + K_1(\epsilon)\mathbf{y}^{(1)} + \ldots + K_{v-1}(\epsilon)\mathbf{y}^{(v-1)}$ (where $K_0(\epsilon), \ldots, K_{v-1}(\epsilon)$ are matrices of dimension $m \times p$)—and then constructing a proper implementation.

Proof:

We shall prove that, for the given ball of initial conditions, a family of proper compensaters $C(s, \epsilon)$ can be designed so that the actuator does not saturate, and further the closed-loop system without saturation is exponentially stable. We first note that, as long as the compensator permits a proper state-space implementation and the system operates in the linear regime, the additive contribution of the compensator's initial condition on the input can be made arbitrarily small through pre- and post-scaling of the compensator by a large gain and its inverse (see Figure 2.1). Thus, WLOG, we seek to verify that $||\mathbf{u}||_\infty < 1$ for the ball of initial states and assuming null compensator initial conditions. To do so, we will design a compensator of the architecture shown in Figure 2.1 that achieves the design goals, and then note a proper implementation.

To do this, let $\tilde{P}$ be any asymptotically stable LTI system of the following form:

$$\begin{bmatrix} \mathbf{y}_{\tilde{P}}^{(1)} \\ \mathbf{y}_{\tilde{P}}^{(2)} \\ \vdots \\ \mathbf{y}_{\tilde{P}}^{(v)} \end{bmatrix} = \begin{bmatrix} & I_m & & \\ & & \ddots & \\ & & & I_m \\ \tilde{Q}_0 & \ldots & \ldots & \tilde{Q}_{v-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\tilde{P}} \\ \mathbf{y}_{\tilde{P}}^{(1)} \\ \vdots \\ \mathbf{y}_{\tilde{P}}^{(v-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} \mathbf{u}_{\tilde{P}}, \tag{2.4}$$

where $\mathbf{u}_{\tilde{P}} \in R^m$ and $\mathbf{y}_{\tilde{P}} \in R^m$ are the input and output to $\tilde{P}$. Notice that $\tilde{P}$ is square-invertible, zero-free, and uniform rank. Let us denote the ∞-norm gain of this plant as q.

Let us first consider pre-compensating the plant G using $\tilde{P}$, and then using feedback of the first v derivatives of the output along with the states of the precompensator (see Figure 2.1). That is, upon precompensation with $\tilde{P}$, we consider using a feedback controller of the form $\mathbf{u}_{\tilde{P}} = \sum_{i=0}^{v-1} K_i \mathbf{y}^{(i)} + \sum_{i=0}^{v-1} \tilde{K}_i \mathbf{y}_{\tilde{P}}^{(i)}$, where we have presciently used the notation K_i for the output-derivative feedbacks since these will turn out to be the gains in the conpensator diagrammed in Figure 2.1d, and where we suppress the dependence on ϵ in our notation for the sake of clarity. For convenience, let us define $K = \begin{bmatrix} K_0 & \dots & K_{v-1} \end{bmatrix}$, $\tilde{K} = \begin{bmatrix} \tilde{K}_0 & \dots & \tilde{K}_{v-1} \end{bmatrix}$, $\mathbf{y}(ext) = \begin{bmatrix} \mathbf{y} \\ \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(v-1)} \end{bmatrix}$, and $\mathbf{y}_{\tilde{P}}(ext) = \begin{bmatrix} \mathbf{y} \\ \mathbf{y}_{\tilde{P}}^{(1)} \\ \vdots \\ \mathbf{y}_{\tilde{P}}^{(v-1)} \end{bmatrix}$. In this notation, the controller becomes $\mathbf{u}_{\tilde{P}} = \begin{bmatrix} K & \tilde{K} \end{bmatrix} \begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$.

We claim that such a controller can be designed, so that 1) the closed-loop system is asymptotically stable, 2) $||u_{\tilde{P}}||_\infty \leq \epsilon$ for the given ball of plant initial conditions and any $0 < \epsilon \leq \frac{0.9}{q}$, and 3) the controller gains K and $\tilde{K}$ are $\mathcal{O}(\epsilon)$. To see why, first note that, based on the fact that the relative degree of the precompensator equals the observability index, the state of the pre-compensated system $\widehat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$ is a linear function of $\begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$. In particular, it is automatic that $\mathbf{x} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix} = Z \begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$, where Z has the form $\begin{bmatrix} Z_1 & Z_2 \\ 0 & I \end{bmatrix}$, see Ding and Pearson's development [11] for the method of construction. Next, from Lemma 2, we see that a low-gain full state-feedback controller $\widehat{K}(\epsilon)$ of order ϵ can be developed for the precompensated plant, that stabilizes the plant and also makes the ∞-norm of the input less than ϵ for any $\epsilon > 0$, for the given ball of plant initial conditions. Thus, by applying the feedback $\widehat{K}(\epsilon)Z \begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$, we can meet the three desired objectives.

It remains to be shown that the plant input $\mathbf{u}$ does not saturate upon application of this compensation. To do so, simply note that $||\mathbf{u}||_\infty \leq q||u_{\tilde{P}}||_\infty \leq 0.9$.

We can absorb the feedback of $\mathbf{y}_{\tilde{P}}(ext)$ into the pre-compensator, so that we obtain a control scheme comprising a precompensator P with dynamics

$$\begin{bmatrix} \mathbf{y}_P^{(1)} \\ \mathbf{y}_P^{(2)} \\ \vdots \\ \mathbf{y}_P^{(v)} \end{bmatrix} = \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & & I_m \\ \tilde{Q}_0 + \tilde{K}_0 & \dots & \dots & \tilde{Q}_{v-1} + \tilde{K}_{v-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_P \\ \mathbf{y}_P^{(1)} \\ \vdots \\ \mathbf{y}_P^{(v-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} \mathbf{u}_P \tag{2.5}$$

together with feedback $\mathbf{u}_P = \sum_{i=0}^{v-1} K_i \mathbf{y}^{(i)}$.

Finally, exactly analogously to the design method in [11]. we see automatically that the transfer function from $\mathbf{y}$ to $\mathbf{u}$ is in fact proper, and so the design admits a proper state-space implementation. Through appropriate scaling of the compensator, we thus see that saturation is avoided for the ball of plant and compensator initial conditions, while the dynamics without saturation are exponentially stable. Thus, semi-global stabilization has been verified.

We have given an alternative low-gain controller design for semi-global stabilization under saturation. It is worth stressing that the crux of the design is the ability to construct the plant's full state using only output derivatives, upon adequate dynamic precompensation. Let us now briefly conceptualize the connections of our design to observer-based designs and zero-cancellation ideas.

Remark: By choosing the $\tilde{Q}_i$ appropriately, we can set the gain q of the precompensator $\tilde{P}$ to an arbitrary value. Appropriate selection of the precompensator can potentially facilitate selection of more numerically-stable feedback gains, by permitting a larger input prior to the precompensator. We leave a careful analysis to future work.

2.2.1 Conceptual Connection with Observer-Based Designs

We have used a precompensator-plus-static-output-feedback representation for a class of feedback controllers to design low-gain stabilizers for LTI plants with actuator saturation, as an alternative to the traditional observer-based architecture for design. The proof of Theorem 1 makes clear the essence of our approach for design: by viewing the controller as comprising an arbitrary dynamic precompensator together with static feedback, we see that the full state of an LTI system can be obtained as a *static* mapping of output derivatives together with the precompensator state variables. This observation yields a design strategy where a dynamic precompensator's impulse response is designed followed by low-gain static state feedback, with the goal of ensuring that the output of their cascade is small (for the given ball of initial conditions). This is a different viewpoint from the traditional one in limited-actuation output-feedback design [2, 3, 6], where actuation capabilities are divided between observation and state-feedback tasks. We believe that this alternative viewpoint can inform design in such domains as decentralized control.

2.3 A Compensator that Exploits Time-Scale Structure

Our philosophy for low-gain control using a precomensation-plus-feedback architecture also permits construction of stabilizers that exploit time-scale structure in the plant. Specifically, we here demonstrate design of precompensators for semi-global stabilization of the plant $\mathcal{G}$, that are generally lower-order than those in Section 2 because they exploit time-scale separation in the plant. Conceptually, when stabilization under saturation is the goal, low-gain state feedback only need be provided for the plant dynamics associated with jw-axis eigenvalues (see e.g. [6] for use of this idea in observer-based designs). In the case where these eigenvalues are at the origin, the corresponding dynamics are in fact the slow dynamics of the system. Thus, through time-scale separation, we can design precompensation together with feedback so as to stabilize the *slow* dynamics under actuator saturation. The use of time-scale separation ideas to reduce the pre-compensator's order becomes rather intricate, and so we illustrate the design only for SISO plants for the sake of clarity. We shall use standard singular-perturbation notions to prove the result. Here is a formal statement:

Theorem 2.4. *Consider a plant $\mathcal{G}$ (as specified in Equation 1) that is SISO, and has q poles at the origin. This plant can be semi-globally stabilized under actuator saturation using a proper dynamic compensator of order q.*

Proof:

We shall prove that, for any given ball of plant and controller initial conditions, there exists a proper compensator of order q such that 1) the linear dynamics of the closed-loop is stable and 2) actuator saturation does not occur.

To this end, let us begin by denoting the the transfer function of the plant's linear dynamics by $G(s)$. We note that the transfer function can be written as $G(s) = \frac{b_m s^m + \ldots + b_0}{s^q(s^{n-q} + a_{n-q-1}s^{n-q-1} + \ldots + a_0)}$, where m is the number of plant zeros. We find it easiest to conceptualize the compensator as comprising a zero-free dynamic precompensator of order q, together with a feedback of the output y and its first $q-1$ derivatives, as shown in Figure 2.1. We choose the precompensator to be *any* stable system of this form. We denote the precompensator's transfer function by $C_p(s) = \frac{1}{(s^q + c_{q-1}s^{q-1} + \ldots + c_0)}$, and the feedback controller by $K(s) = k_{q-1}s^{q-1} + \ldots + k_0$. We note the entire compensator $K(s)C_p(s)$ has order q and is proper.

With a little algebra, we find that the characteristic polynomial of the closed-loop system is $p(s) = s^q(s^{n-q} + a_{n-q-1}s^{n-q-1} + \ldots + a_0)(s^q + c_{q-1}s^{q-1} + \ldots + c_0) + (b_m s^m + \ldots + b_0)(k_{q-1}s^{q-1} + \ldots + k_0)$. We note that the polynomial $p_f(s) = (s^{n-q} + a_{n-q-1}s^{n-q-1} + \ldots + a_0)(s^q + c_{q-1}s^{q-1} + \ldots + c_0)$ has roots in the OLHP, by assumption, for the chosen stable pre-compensator $C_p(s)$.

Let us now consider a family of multiple derivative controllers, parameterized by a low-gain parameter $\epsilon > 0$. In particular, let us consider a controller with $k_i = \frac{a_0 c_0}{b_0}\gamma_i \epsilon^{q-i}$, where $s^q + \gamma_{q-1}s^{q-1} + \ldots + \gamma_0$ is a stable polynomial with roots $\lambda_1, \ldots, \lambda_q$, and ϵ is a low-gain parameter.

Next, we will verify that the characteristic polynomial has n roots that are within $\mathcal{O}(\epsilon)$ of the roots of $p_f(s)$, while the remaining q roots are within $\mathcal{O}(\epsilon^2)$ of $\epsilon\lambda_1, \ldots, \epsilon\lambda_q$. To prove this, notice first that $p(s) = s^q p_f(s) + (b_m s^m + \ldots + b_0)(\gamma_{q-1}\epsilon s^{q-1} + \ldots + \gamma_0 \epsilon^q)\frac{a_0 c_0}{b_0}$. Noting that the entire second term in this expression is $\mathcal{O}(\epsilon)$, we see that the roots of $p(s)$ are $\mathcal{O}(\epsilon)$ perturbations of the roots of $s^q p_f(s)$. Thus, we see that n roots are within $\mathcal{O}(\epsilon)$ of the roots of $p_f(s)$, while the remaining are within $\mathcal{O}(\epsilon)$ of the origin.

To continue, let us consider the change of variables $\overline{s} = \frac{\epsilon}{s}$. Substituting into the closed-loop characteristic polynomial, we find that $p(\overline{s}) = (\frac{\epsilon}{\overline{s}})^q p_f(\overline{s}) + \epsilon^q(b_m \frac{\epsilon^m}{\overline{s}^m} + \ldots + b_0)(\frac{\gamma_{q-1}}{\overline{s}^{q-1}} + \ldots + \gamma_0)\frac{a_0 c_0}{b_0}$.

Scaling the expression by $\frac{\overline{s}^{n+q}}{a_0 c_0}$, we obtain that the expression $p(\overline{s}) = 0$ is the following degree-$(n+q)$ polynomial equation in $\overline{s}$: $\epsilon^q(\gamma_0 \overline{s}^{n+q} + \ldots + \gamma_{q-1}\overline{s}^{n+1}) + \epsilon^q \overline{s}^n + r(\overline{s}) = 0$, where $r(\overline{s})$ is a polynomial in $\overline{s}$ of degree no more that $\overline{s}^{n+q-1}$ with each term scaled by a coefficient of order ϵ^{q+1} or smaller. Thus, dividing by ϵ^q, we find that the solutions $\overline{s}$ to the equation are within $\mathcal{O}(\epsilon)$ of the solutions to $\gamma_0 \overline{s}^{n+q} + \ldots + \gamma_{q-1}\overline{s}^{n+1} + \overline{s}^n = 0$. However, the roots of this equation are precisely $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_q}$, as well as 0 repeated n times. Noting that $s = \frac{\epsilon}{\overline{s}}$, we thus recover that q roots of the characteristic polynomial are within $\mathcal{O}(\epsilon^2)$ of $\epsilon\lambda_1, \ldots, \epsilon\lambda_q$. Thus, we have characterized all the poles of the closed-loop system. We notice that all the poles are guaranteed to be within the OLHP.

Now consider the response for a ball of initial conditions $\mathcal{W}$. As in the proof of Theorem 1, we notice that the initial state of the precompensator is of no concern in terms of causing saturation, since the precompensator can be pre- and post-scaled by an arbitrary positive constant. Thus, WLOG, let us consider selecting among the family of compensators, to avoid saturation for a given ball of plant initial conditions and assuming zero precompensator initial conditions. Through consideration of the closed-loop dynamics associated with the slow eigenvalues $(\epsilon\lambda_1, \ldots, \epsilon\lambda_q)$, we recover immediately (see the proof of Lemma 1 in [2]) that, for any specified ball of initial conditions, $||y^{(i)}(t)||_\infty$, is at most of order $\frac{1}{\epsilon^{q-1-i}}$ for $i = 1, \ldots, q-1$. Thus, from the expression for the feedback controller, we find that the maximum value of the precompensator input $\overline{u}$ is $\mathcal{O}(\epsilon)$, say $v_1\epsilon + \mathcal{O}(\epsilon^2)$ for the given ball of plant initial conditions. Furthermore, the stable precompensator imparts a finite gain, say v_2, so the maximum value of $u(t)$ is $v_1 v_2 \epsilon + \mathcal{O}(\epsilon^2)$. Thus, for any given ball of initial conditions, we can choose ϵ small enough so that actuator saturation does not occur. Since actuator saturation is avoided and the closed-loop poles are in the OLHP, stability is proved.

Conceptually, the reduction in the controller order permitted by Theorem 2 is founded on focusing the control effort on only the slow dynamics of the system. That is, the controller is designed only to place the eigenvalues at the origin at desired locations (that are linear with respect to the low-gain parameter ϵ); simply using small gains enforces that the remaining eigenvalues remain far in the OLHP. Thus, one only needs to add precompensation to permit estimation of the part of the state associated with the slow dynamics. In this way, stability can be guaranteed and saturation avoided, without requiring as much precompensation as would be needed to estimate the whole state.

We notice that the time-scale-based design is aligned with the broad philosophy of our alternative low-gain design, in the sense that it provides freedom in compensator design. In particular, as with the design in Section 2, we notice that *any* stable precompensator can be used for the time-scale-exploiting design, and further design of feedback component in the architecture only requires knowlege of the DC gain of the plant.

2.4 Example

In this example, we demonstrate the design of a low-gain controller that semi-globally stabilizes the following plant:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma(\mathbf{u}) \tag{2.6}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}. \tag{2.7}$$

Specifically, we show that the compensator of the form $\mathbf{u}(t)^{(v)} = \sum_0^{v-1} \mathbf{A}_i \mathbf{u}(t)^{(i)} + \sum_0^{(v-1)} \mathbf{B}_i \mathbf{y}(t)^{(i)}$, where v is the observability index of the plant, can stabilize the plant under saturation. As developed in the article, the design is achieved by first designing a pre-compensator together with output feedback control law, and then implementing the controller in the proper feedback representation above. We shall use the notation from the above development in our illustration.

Let us begin with the precompensator-plus-output-feedback design. To begin, we notice that the observability index of this system is 2. As per the proof of Theorem 1, let us thus choose $\tilde{P}$ to be

$$\begin{bmatrix} \dot{\mathbf{y}}_{\tilde{P}} \\ \ddot{\mathbf{y}}_{\tilde{P}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\tilde{P}} \\ \dot{\mathbf{y}}_{\tilde{P}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_{\tilde{P}} \tag{2.8}$$

The eigenvalues of this system are located at $-1/2 \pm \sqrt{3}/2i$, hence $\tilde{P}$ is clearly asymptotically stable.

Now let us pre-compensate the plant (Equation 2.6) using $\tilde{P}$ and design the feedback controller of the form

$$\mathbf{u}_{\tilde{P}} = \tilde{K}_0(\epsilon)\mathbf{y}_{\tilde{P}} + \tilde{K}_1(\epsilon)\dot{\mathbf{y}}_{\tilde{P}} + K_0(\epsilon)\mathbf{y} + K_1(\epsilon)\dot{\mathbf{y}} \tag{2.9}$$

To do so, we first recover the state of the entire system including the pre-compensator from the outputs $\mathbf{y}_{\tilde{P}}, \dot{\mathbf{y}}_{\tilde{P}}, \mathbf{y}, \dot{\mathbf{y}}$ through linear transformation, and then apply the low-gain state feedback design (see [2]) to shift eigenvalues of the whole system left by $-\epsilon$. Doing so, we have

$$\tilde{K}_0(\epsilon) = \begin{bmatrix} -0.42\epsilon^5 + 0.30\epsilon^4 + 1.8\epsilon^3 - 4.2\epsilon^2 - 0.90\epsilon & 0.082\epsilon^5 - 1.0\epsilon^4 + 5.2\epsilon^3 - 11.4\epsilon^2 + 5.8\epsilon \\ -0.11\epsilon^5 + 0.28\epsilon^4 - 0.55\epsilon^3 - 6.0\epsilon^2 - 2.2\epsilon & 0.024\epsilon^5 - 0.3\epsilon^4 + 1.5\epsilon^3 - 4.3\epsilon^2 - 0.33\epsilon \end{bmatrix},$$

$$\tilde{K}_1(\epsilon) = \begin{bmatrix} -0.0084\epsilon^5 + 0.13\epsilon^4 - 0.66\epsilon^3 + 0.74\epsilon^2 - 5.3\epsilon & 0.029\epsilon^5 - 0.47\epsilon^4 + 2.3\epsilon^3 - 2.5\epsilon^2 + 1.0\epsilon \\ -0.0024\epsilon^5 + 0.039\epsilon^4 - 0.19\epsilon^3 + 0.21\epsilon^2 - 0.95\epsilon & 0.0084\epsilon^5 - 0.13\epsilon^4 + 0.66\epsilon^3 - 0.74\epsilon^2 - 1.70\epsilon \end{bmatrix},$$

$$K_0(\epsilon) = \begin{bmatrix} -0.30\epsilon^5 - 1.2\epsilon^4 - 1.2\epsilon^3 - 0.91\epsilon^2 & -0.71\epsilon^5 + 4\epsilon^4 - 15.7\epsilon^3 + 34\epsilon^2 - 23\epsilon \\ -0.088\epsilon^5 - 0.35\epsilon^4 - 0.35\epsilon^2 - 0.26\epsilon^2 & -1.0\epsilon^5 + 1.18\epsilon^4 - 4.6\epsilon^3 + 10\epsilon^2 - 6.7\epsilon \end{bmatrix}$$

and

$$K_1(\epsilon) = \begin{bmatrix} 0.39\epsilon^5 - 0.0\epsilon^4 - 3.3\epsilon^3 - 2.5\epsilon^2 - 1.8\epsilon & 0 \\ 0.091\epsilon^5 + 0.020\epsilon^4 - 0.95\epsilon^3 - 0.72\epsilon^2 - 0.53\epsilon & 0 \end{bmatrix}.$$

Hence, the control scheme can be viewed as comprising a precompensator P:

$$\begin{bmatrix} \dot{\mathbf{y}}_P \\ \ddot{\mathbf{y}}_P \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} + \tilde{K}_0(\epsilon) & -\mathbf{I} + \tilde{K}_1(\epsilon) \end{bmatrix} \begin{bmatrix} \mathbf{y}_P \\ \dot{\mathbf{y}}_P \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_P \tag{2.10}$$

and the feedback flow $\mathbf{u}_P = K_0(\epsilon)\mathbf{y} + K_1(\epsilon)\dot{\mathbf{y}}$.

Finally, the above procedure leads to the proper feedback compensator design

$$\mathbf{u}(t)^{(2)} = (-\mathbf{I} + \tilde{K}_0(\epsilon))\mathbf{u}(t) + (-\mathbf{I} + \tilde{K}_1(\epsilon))\mathbf{u}^{(1)}(t) + K_0(\epsilon)\mathbf{y}(t) + K_1(\epsilon)\mathbf{y}^{(1)}(t) \tag{2.11}$$

Now let us show how ϵ can be chosen. In particular, consider the case where initial conditions of the plant and compensator are in a ball $\mathcal{W}$ with infinity-norm radius 1, i.e., where each initial condition has a magnitude less than or equal to 1. We find that $\epsilon^*(\mathcal{W}) \approx 0.5$ through an exhaustive search. Thus ϵ can be chosen between 0 and 0.5. Figure 2.2 and Figure 2.3 show the trajectories of inputs under two intitial conditions.

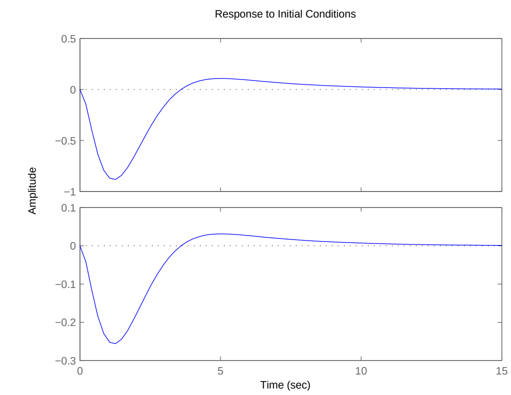

Fig. 2.2: $\epsilon = 0.5$ a)trajectory of input u_1; b)trajectory of input u_2.

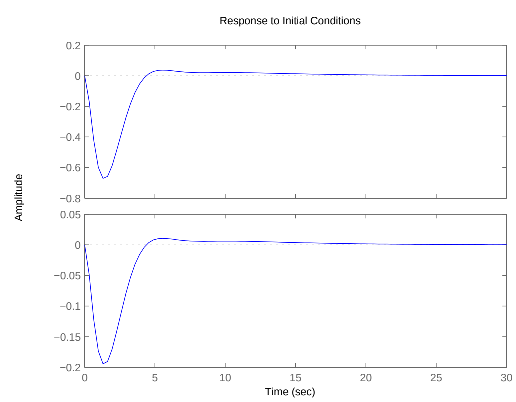

Fig. 2.3: $\epsilon = 0.25$ a)trajectory of input u_1; b)trajectory of input u_2.

3. DECENTRALIZED CONTROL OF DISCRETE-TIME LINEAR TIME INVARIANT SYSTEMS WITH INPUT SATURATION

We study decentralized stabilization of discrete-time linear time invariant (LTI) systems subject to actuator saturation, using LTI controllers [19]. The requirement of stabilization under both saturation constraints and decentralization impose obvious necessary conditions on the open-loop plant, namely that its eigenvalues are in the closed unit disk and further that the eigenvalues on the unit circle are not decentralized fixed modes. The key contribution of this work is to provide a broad sufficient condition for decentralized stabilization under saturation. Specifically, we show through an iterative argument that stabilization is possible whenever 1) the open-loop eigenvalues are in the closed unit disk, 2) the eigenvalues on the unit circle are not decentralized fixed modes, and 3) these eigenvalues on the unit circle have algebraic multiplicity of 1.

3.1 Introduction

The result presented here contributes to our ongoing study of the stabilization of decentralized systems subject to actuator saturation. The eventual goal of this study is the *design* of controllers for saturating decentralized systems that achieve not only stabilization but also high performance. As a first step toward this design goal, we are currently looking for tight conditions on a decentralized plant with input saturation, for the *existence* of stabilizing controllers. Even this check for the existence of stabilizing controllers turns out be extremely intricate: we have yet to obtain necessary and sufficient conditions for stabilization, but have obtained a broad sufficient condition in our earlier work [16]. This article further contributes to the study of the existence of stabilizing controllers, by describing a analogous sufficient condition for discrete-time decentralized plants.

To motivate and introduce the main result in the article, let us briefly review foundational studies on both decentralized control and saturating control systems. We recall that a necessary and sufficient condition for stabilization of a decentralized system using LTI state-space controllers is given in Wang and Davison's classical work [22]. They obtain that stabilization is possible if and only if all *decentralized fixed modes* of a plant are in the OLHP, and give specifications of and methods for finding these decentralized fixed modes. Numerous further characterizations of decentralized stabilization (and fixed modes) have been given, see for instance the work of Corfmat and Morse [14]. In complement, for centralized control systems subject to actuator saturation, not only conditions for stabilization but also practical designs have been obtained, using the low-gain and low-high-gain methodology. For a background on the results for centralized systems subject to input saturation we refer to two special issues [13, 15]. Of importance here, we recall that a necessary and sufficient condition for *semi-global* stabilization of LTI plants with actuator saturation is that their open-loop poles are in the CLHP. Combining this observation with Wang and Davison's result, one might postulate that stabilization of a saturating linear decentralized control system is possible if and only of 1) the open-loop plant poles are in the CLHP (respectively, closed unit disk, for discrete-time systems), and 2) the poles on the imaginary axis (respectively, unit circle) are not decentralized fixed modes. The necessity of the two requirements is immediate, but we have not yet been able to determine whether the requirements are also sufficient. As a first step for continuous-time plants, we showed in [16] that decentralized stabilization under saturation is possible when 1) the plant's open-loop poles are in the CLHP with imaginary axis poles non-repeated, and 2) the imaginary axis poles are not decentralized fixed modes. Here, we develop an analogous result for discrete-time plants, in particular showing that decentralized stabilization under saturation is possible if 1) the plant's open-loop poles are in the closed unit disk with unit-circle poles non-repeated, and 2) the unit-circle poles are not decentralized fixed modes.

3.2 Problem formulation

Consider the LTI discrete-time systems subject to actuator saturation,

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{\nu} B_i sat(u_i(k)) \\ y_i(k) \quad = C_i x(k), \qquad i = 1, \ldots, \nu, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is state, $u_i \in \mathbb{R}^{m_i}$, $i = 1, \ldots, \nu$ are control inputs, $y_i \in \mathbb{R}^{p_i}$, $i = 1, \ldots, \nu$ are measured outputs, and 'sat' denotes the standard saturation element.

Here we are looking for ν controllers of the form,

$$\Sigma_i : \begin{cases} z_i(k+1) = K_i z_i(k) + L_i y_i(k), \quad z_i \in \mathbb{R}^{s_i} \\ u_i(k+1) = M_i z_i(k) + N_i y_i(k). \end{cases} \tag{3.2}$$

Problem 1. *Let the system* (4.1) *be given. The* ***semi-globally stabilization problem via decentralized control*** *is said to be solvable if for all compact sets* $\mathcal{W}$ *and* $\mathcal{S}_1, \ldots, \mathcal{S}_\nu$ *there exists* ν *controllers of the form* (4.2) *such that the closed loop system is asymptotically stable with the set*

$$\mathcal{W} \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_\nu$$

contained in the domain of attraction.

The main objective of this research is to develop necessary conditions and sufficient conditions such that the semi-global stabilization problem via decentralized control is solvable. This objective has not yet been achieved. However, we obtain necessary conditions as well as sufficient conditions which are quite close.

3.3 Review of discrete-time LTI decentralized systems and stabilization

Before we tackle the aforementioned problem in Section 3.2, let us first review the necessary and sufficient conditions for the decentralized stabilization of the linearized model of the given system Σ,

$$\bar{\Sigma} : \begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{\nu} B_i u_i(k) \\ y_i(k) \quad = C_i x(k), \qquad i = 1, \ldots, \nu, \end{cases} \tag{3.3}$$

The decentralized stabilization problem for $\bar{\Sigma}$ is to find LTI dynamic controllers Σ_i, $i = 1, \ldots, \nu$, of the form (4.2) such that the poles of the closed loop system are in the desired locations in the open unit disc.

Given system $\bar{\Sigma}$ and controllers Σ_i, defined by (3.3) and (4.2) respectively, let us first define the following block diagonal matrices in order to provide an easier bookkeeping:

$$B = [B_1, \ldots, B_\nu], \qquad C = [C_1, \ldots, C_\nu]$$

$$K = [K_1, \ldots, K_\nu], \qquad L = [L_1, \ldots, L_\nu]$$

$$M = [M_1, \ldots, M_\nu], \qquad N = [N_1, \ldots, N_\nu]$$

Definition 3.1. *Consider system $\bar{\Sigma}$, $\lambda \in \mathbb{C}$ is called a decentralized fixed mode if for all block* *diagonal matrices H we have*

$$\det(\lambda I - A - BHC) = 0$$

Efficectively we look at eigenvalues that can be moved by static decentralized controllers. However, it is known that if we cannot move an eigenvalue by static decentralized controllers then we cannot move the eigenvalue by dynamic decentralized controllers either.

Lemma 3.2. *Necessary and sufficient condition for the existence of a decentralized feedback control law for the system $\bar{\Sigma}$ such that the closed loop system is asymptotically stable is that all the fixed modes of the system be asymptotically stable(in the unit disc).*

Proof: We first establish necessity.

Assume local controllers Σ_i together stabilize $\bar{\Sigma}$ then for any $|\lambda| \geq 1$ there exists a δ such that $(\lambda+\delta)I - K$ is invertible and the closed loop system replacing K with $K - \delta I$ is still asymptotically stable. This choice is possible because if $\lambda I - K$ is invertible obviously we can choose $\delta = 0$. If $\lambda I - K$ is not invertible, by small enough choice of δ we can make sure that $(\lambda+\delta)I - K$ is invertible and the closed loop system replacing K with $K - \delta I$ is still asymptotically stable. Since the closed loop system with $K - \delta I$ is still asymptotically stable, it can not have a pole in λ. So

$$\det(\lambda I - A - B[M(\lambda I - (K - \delta I))^{-1}L + N]C) \neq 0$$

Hence the block diagonal matrix

$$S = M(\lambda I - (K - \delta I))^{-1}L + N$$

has the property that

$$\det(\lambda I - A - BSC) \neq 0$$

which implies that λ is not a fixed mode. Since this arguement is true for any λ on or outside the unit disc, this implies that all the fixed modes must be inside the unit disc. This proves the necessity of the Lemma 4.2.

Next, we establish sufficiency. The papers [14, 22] showed that if the decentralized fixed modes of a strongly connected system are stable, we can find a stabilizing controller for the system. However, these papers are based on continuous-time results. For completeness we present the proof for discrete-time which is a straightforward modification of [22]. We first claim that decentralized fixed modes are invariant under preliminary output injection. But this is obvious from our necessity proof since a trivial modification shows that no dynamic controller can move a fixed mode nor can it introduce additional fixed modes. To prove that we can actually stabilize the system, we use a recursive argument. Assume the system has an unstable eigenvalue in μ. Since μ is not a fixed mode there exists N_i such that

$$A + \sum_{i=1}^{\nu} B_i N_i C_i$$

no longer has an eigenvalue in μ. Let k be the smallest integer such that

$$A + \sum_{i=1}^{k} B_i N_i C_i$$

no longer has an eigenvalue in μ. Then for the system

$$\left(A + \sum_{i=1}^{k-1} B_i N_i C_i, B_k, C_k \right)$$

the eigenvalue μ is both observable and controllable. But this implies that there exists a dynamic controller which moves this eigenvalue in the open unit disc without introducing new unstable eigenvalues. Through a recursion, we can move all eigenvalues one-by-one in the open unit disc and in this way find a decentralized controller which stabilizes the system. This proves the sufficiency of the lemma 4.2.

3.4 Main Result

In this section, we present the main results of this paper.

Theorem 3.3. *Consider the system Σ. There exists nonnegative integers $s_1, \cdots, s_\nu$ such that for any given collection of compact sets $W \subset \mathbb{R}^n$ and $S_i \subset \mathbb{R}^{s_i}$, $i = 1, \cdots, \nu$, there exists ν controllers of the form* (4.2) *such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes $W \times S_1 \times \cdots \times S_\nu$ only if*

- *All fixed modes are in the open unit disc.*
- *All eigenvalues of A are in the closed unit disc.*

Proof: There exists an open neighborhood containing the origin where the interconnection of Σ with the controllers Σ_i is identical to the interconnection of $\bar{\Sigma}$ with the controllers Σ_i. Hence (local) asymptotic stability of one closed loop system is equivalent to asymptotic stability of the other closed loop system. But then it is obvious from Lemma 4.2 that the first item of Theorem 3.3 is necessary for the existence of controllers of the form (4.2) for $\bar{\Sigma}$ such that the origin of the resulting closed loop system is asymptotically stable.

To prove the necessity of the second item of Theorem 3.3, assume that λ is an eigenvalue of A outside the unit disc with associated left eigenvector p. We obtain:

$$px(k+1) = \lambda px(k) + v(k)$$

where

$$v(k) := \sum_{i=1}^{\nu} pB_i sat(u_i(k)).$$

Because of the saturation elements, there exists an $\tilde{M} > 0$ such that $|v(k)| \leq \tilde{M}$ for all $k \geq 0$. But then we have

$$px(k) = \lambda^k px(0) + \sum_{i=0}^{k-1} \lambda^{k-1-i} v(i) = \lambda^k (px(0) + S_k), \tag{3.4}$$

where $S_k = \sum_{i=0}^{k-1} \frac{v(i)}{\lambda^{i+1}}$. We find that

$$|S_k| \leq \tilde{M} \sum_{i=1}^{k} \frac{1}{|\lambda|^i} = \tilde{M} \cdot \frac{1 - \frac{1}{|\lambda|^k}}{|\lambda| - 1} < \frac{\tilde{M}}{|\lambda| - 1}$$

and then from (3.4) we find

$$|px(k)| > |\lambda|^k \left(|px(0)| - \tfrac{\tilde{M}}{|\lambda|-1} \right) \;\; \forall k \geq 1.$$

Hence $|px(k)|$ does not converge to zero independent of our choice for a controller if we choose the initial condition $x(0)$ such that $|px(0)| > \frac{\tilde{M}}{|\lambda|-1}$ because of the fact that $|\lambda| > 1$. However, the system was semi-globally stabilizable and hence there exists a controller which contains this initial condition in its domain of attraction and hence $|px(k)| \to 0$ which yields a contradiction. This proves the second item of Theorem 3.3.

We now proceed to the next theorem which gives a sufficient condition for semi-global stabilizability of (4.1) when the set of controllers given by (4.2) are utilized.

Theorem 3.4. *Consider the system Σ. There exists nonnegative integers $s_1, \cdots, s_\nu$ such that for any given collection of compact sets $W \subset \mathbb{R}^n$ and $S_i \subset \mathbb{R}^{s_i}$, $i = 1, \cdots, \nu$, there exists ν controllers of the form (4.2) such that the origin of the resulting closed loop system is asymptotically stable and the domain of attraction includes $W \times S_1 \times \cdots \times S_\nu$ if*

- *All fixed modes are in the open unit disc,*
- *All eigenvalues of A are in the closed unit disc with those eigenvalues on the unit disc having algebraic multiplicity equal to one.*

To prove this theorem we will exploit the following lemma which follows directly from classical results of eigenvalues and eigenvectors and the results of perturbations of the matrix on those eigenvalues and eigenvectors.

Lemma 3.5. *Let $A_\delta \in \mathbb{R}^{n\times n}$ be a sequence of matrices parametrized by δ and let a matrix $A \in \mathbb{R}^{n\times n}$ be such that $A_\delta \to A$ as $\delta \to 0$. Assume that A is a matrix with all eigenvalues in the closed unit disc and with p eigenvalues on the unit disc with all of them having multiplicity 1. Also assume that A_δ has all its eigenvalues in the closed unit disc. Let matrix $P > 0$ be such that $A'PA - P \leq 0$ is satisfied. Then for small $\delta > 0$ there exists a family of matrices $P_\delta > 0$ such that*

$$A'_\delta P_\delta A_\delta - P_\delta \leq 0$$

and $P_\delta \to P$ as $\delta \to 0$.

Proof: We first observe that there exists a matrix S such that

$$S^{-1}AS = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

where all eigenvalues of A_{11} are on the unit circle while the eigenvalues of A_{22} are in the open unit disc. Since $A_\delta \to A$ and the eigenvalues of A_{11} and A_{22} are distinct, there exists a parametrized matrix S_δ such that for sufficiently small δ

$$S_\delta^{-1} A_\delta S_\delta = \begin{pmatrix} A_{11,\delta} & 0 \\ 0 & A_{22,\delta} \end{pmatrix}$$

where $S_\delta \to S$, $A_{11,\delta} \to A_{11}$ and $A_{22,\delta} \to A_{22}$ as $\delta \to 0$.

Given a matrix $P > 0$ such that $A'PA - P \leq 0$. Let us define

$$\bar{P} = S'PS = \begin{pmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}'_{12} & \bar{P}_{22} \end{pmatrix}$$

with this definition we have

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \leq 0. \tag{3.5}$$

Next given an eigenvector x_1 of A_{11}, i.e. $A_{11}x_1 = \lambda x_1$ with $|\lambda| = 1$, we have

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}^* \left[\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Using (3.5), the above implies that

$$\left[\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Since all the eigenvalues of $A_{11} \in \mathbb{R}^{p \times p}$ are distinct we find that the eigenvectors of A_{11} span $\mathbb{R}^p$ and hence

$$\left[\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} \right] \begin{pmatrix} I \\ 0 \end{pmatrix} = 0.$$

This results in

$$\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} \bar{P} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} - \bar{P} = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} \leq 0.$$

This implies that $A'_{11}\bar{P}_{12}A_{22} - \bar{P}_{12} = 0$ and since eigenvalues of A_{11} are on the unit disc and eigenvalues of A_{22} are inside the unit disc, we find that $\bar{P}_{12} = 0$ because

$$A'_{11}\bar{P}_{12}A_{22} = \bar{P}_{12} \Rightarrow (A'_{11})^k \bar{P}_{12} A_{22}^k = \bar{P}_{12}$$

where k is an arbitrary positive integer. Note that $(A'_{11})^k$ remains bounded while $A_{22}^k \to 0$ as $k \to \infty$, This means that for $k \to \infty$, $\bar{P}_{12} \to 0$ and because $\bar{P}_{12}$ is independent of k, we find that $\bar{P}_{12} = 0$. Next, since A_{22} has all its eigenvalues in the open unit disc, there exists a parametrized matrix $P_{\delta,22}$ such that for δ small enough

$$A'_{\delta,22} P_{\delta,22} A_{\delta,22} - P_{\delta,22} = V \leq 0$$

while $P_{\delta,22} \to P_{22}$ as $\delta \to 0$.

Let $A_{11} = W \Lambda_A W^{-1}$ with Λ_A a diagonal matrix. Because the eigenvectors of A_{11} are distinct and $A_{11,\delta} \to A_{11}$, the eigenvectors of $A_{11,\delta}$ depend continuously on δ for δ small enough and hence there exists a parametrized matrix W_δ such that $W_\delta \to W$ while $A_{11,\delta} = W_\delta \Lambda_{A_\delta} W_\delta^{-1}$ with Λ_{A_δ} diagonal. The matrix $\bar{P}_{11}$ satisfies

$$A'_{11} \bar{P}_{11} A_{11} - \bar{P}_{11} = 0.$$

This implies that $\Lambda_P = W^* \bar{P}_{11} W$ satisfies

$$\Lambda_A^* \Lambda_P \Lambda_A - \Lambda_P = 0.$$

The above equation then shows that Λ_P is a diagonal matrix. We know that

$$\Lambda_{A_\delta} \to \Lambda_A.$$

We know that Λ_{A_δ} is a diagonal matrix the diagonal elements of which have magnitude less or equal to one while Λ_P is a positive definite diagonal matrix.

Using this, it can be verified that we have

$$\Lambda_{A_\delta}^* \Lambda_P \Lambda_{A_\delta} - \Lambda_P \leq 0$$

We choose $\bar{P}_{11,\delta}$ as

$$\bar{P}_{11,\delta} = (W_\delta^*)^{-1}\Lambda_P(W_\delta)^{-1}$$

We can see that this choice of $\bar{P}_{11,\delta}$ satisfies

$$A'_{11,\delta}\bar{P}_{11,\delta}A_{11,\delta} - \bar{P}_{11,\delta} \leq 0$$

It is easy to see that $\bar{P}_{11,\delta} \to \bar{P}_{11}$ as $\delta \to 0$. Then

$$P_\delta = (S_\delta^{-1})' \begin{pmatrix} \bar{P}_{11,\delta} & 0 \\ 0 & \bar{P}_{22,\delta} \end{pmatrix} S_\delta^{-1}$$

satisfies the condition of the lemma. This completes the proof of Lemma 3.5.

We now show a recursive algorithm that at each step moves at least one eigenvalue on the unit circle in a decentralized fashion while preserving the stability of other modes in the open unit disc in a way that the magnitude of each decentralized feedback control is assured never to exceed $1/n$. The algorithm will consist of at most n steps, and therefore the overall decentralized inputs will not saturate for an appropriate choice of the initial state.

Algorithm:

- **Step 0:** We initialize algorithm at this step. Let $A_0 := A$, $B_{0,i} := B_i$, $C_{0,i} := C_i$, $n_{i,0} := 0$, $N_{i,\varepsilon}^0 := 0$, $i = 1, \ldots, \nu$ and $x_0 := x$. Also let us define $P_0^\varepsilon := \varepsilon P$, where $P > 0$ and satisfies $A'PA - P \leq 0$.

- **Step $\boldsymbol{m}$:** For the system Σ, we want to design ν parametrized decentralized feedback control laws,

$$\Sigma_i^{m,\varepsilon} : \begin{cases} p_i^m(k+1) = K_{i,\varepsilon}^m p_i^m(k) + L_{i,\varepsilon}^m y_i(k) \\ u_i(k) = M_{i,\varepsilon} p_i^m(k) + N_{i,\varepsilon}^m y_i(k) + v_i^m(k). \end{cases}$$

where $p_i^m \in \mathbb{R}^{n_{i,m}}$ and if $n_{i,m} = 0$:

$$\Sigma_i^{m,\varepsilon} : \left\{ \; u_i(k) = N_{i,\varepsilon}^m y_i(k) + v_i^m(k) \right.$$

The closed loop system consisting the decentralized controller and the system Σ can be written as

$$\Sigma_{cl}^{m,\varepsilon} : \begin{cases} x_m(k+1) = A_m^\varepsilon x_m(k) + \sum_{i=1}^{\nu} B_{m,i} v_i^m(k) \\ y_i(k) = C_{m,i} x_m(k), \qquad i = 1, \cdots, \nu \end{cases}$$

where $x_m \in \mathbb{R}^{n_m}$ with $n_m = n + \sum_{i=1}^{\nu} n_{i,m}$ is given by

$$x_m = \begin{pmatrix} x \\ p_1^m \\ \vdots \\ p_\nu^m \end{pmatrix}.$$

We can rewrite u_i as

$$u_i = F_{i,\varepsilon}^m x_m + v_i^m,$$

for some appropriate matrix $F_{i,\varepsilon}^m$.

Our objective here is to design the decentralized stabilizers in such a way that they satisfy the following properties:

1) Matrix A_m^ε has all its eigenvalues in the closed unit disc, and eigenvalues on unit disc are distinct.

2) A_m^ε has less eigenvalues on the unit circle than A_{m-1}^ε.

3) There exists a family of matrices P_m^ε such that $P_m^\varepsilon \to 0$ as $\varepsilon \to 0$ and

$$(A_m^\varepsilon)' P_m^\varepsilon A_m^\varepsilon - P_m^\varepsilon \leq 0$$

Furthermore, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$ and $\nu_i^m = 0$ we have $\|u_i(k)\| \leq \frac{m}{n}$ for all states with $x_m'(k) P_m^\varepsilon x_m(k) \leq n - m + 1$.

- **Terminal Step:** There exists a value for m, say $l \leq n$, such that A_l^ε has all its eigenvalues in the open unit disc, and also property 3 above is satisfied, which means that for ε small enough, $\|u_i\| \leq 1$ for all states with $x_l' P_l^\varepsilon x_l \leq 1$. The decentralized control laws $\Sigma_i^{l,\varepsilon}$, $i = 1, \ldots, l$ together construct our decentralized feedback law for system Σ.

Finally, we show that for an appropriate choice of ε, this recursive algorithm provides a set of decentralized feedbacks which satisfy the requirements of Theorem 3.4. We will first prove properties 1, 2 and 3 listed above by induction. It is easy to see that the initialization step satisfies these properties. We assume that the design in the step m can be done, and then we must show that the design in the step $m + 1$ can be done.

Now assume that we are in step $m + 1$. The closed loop system $\Sigma_{cl}^{m,\varepsilon}$ has properties (1), (2) and (3). Let λ be an eigenvalue on the unit disc of A_m^ε. We know that λ is not a fixed mode of the closed loop system. Thus there exist $\bar{K}_i$ such that

$$A_m^\varepsilon + \sum_{i=1}^{\nu} B_{m,i} \bar{K}_i C_{m,i}$$

has no eigenvalue at λ. Therefore the determinant of the matrix $\lambda I - A_m^\varepsilon - \delta \sum_{i=1}^{\nu} B_{m,i} \bar{K}_i C_{m,i}$, seen as a polynomial in δ, is non-zero for $\delta = 1$, which implies that it is non-zero for almost all $\delta > 0$. This means that for almost all $\delta > 0$,

$$A_m^\varepsilon + \delta \sum_{i=1}^{\nu} B_{m,i} \bar{K}_i C_{m,i}$$

has no eigenvalue at λ. Let j be the largest integer such that

$$A_m^{\varepsilon,\delta} = A_m^\varepsilon + \delta \sum_{i=1}^{j} B_{m,i} \bar{K}_i C_{m,i}$$

has λ as an eigenvalue and the same number of eigenvalues on the unit disc as A_m^ε for small enough δ. This implies that $A_m^{\varepsilon,\delta}$ still has all its eigenvalues in the closed unit disc.

Using Lemma 3.5, we know that there exists a $P_m^{\varepsilon,\delta}$ such that

$$(A_m^{\varepsilon,\delta})' P_m^{\varepsilon,\delta} A_m^{\varepsilon,\delta} - P_m^{\varepsilon,\delta} \leq 0$$

while $P_m^{\varepsilon,\delta} \to P_m^\varepsilon$ as $\delta \to 0$. Hence for small enough δ

$$x_m'(k) P_m^{\varepsilon,\delta} x_m(k) \leq n - m + \frac{1}{2} \Rightarrow x_m'(k) P_m^\varepsilon x_m(k) \leq n - m + 1$$

and also for small enough δ we have

$$\|\delta \bar{K}_i x_m\| \leq \frac{1}{2n} \qquad \forall\, x_m \text{ such that } x_m' P_m^{\varepsilon,\delta} x_m \leq n - m + \frac{1}{2}.$$

We choose $\delta = \delta_\varepsilon$ small enough such that the above two properties hold. Define $K_i^\varepsilon = \delta_\varepsilon \bar{K}_i$, $\bar{P}_m^\varepsilon = \bar{P}_m^{\varepsilon,\delta_\varepsilon}$ and

$$\bar{A}_m^\varepsilon := A_m^\varepsilon + \sum_{i=1}^{j} B_{m,i} K_i^\varepsilon C_{m,i}.$$

By the definition of j, we know that

$$A_m^\varepsilon + \sum_{i=1}^{j+1} B_{m,i} K_i^\varepsilon C_{m,i}$$

either does not have λ as an eigenvalue or has less eigenvalues on the unit disc. This means that

$$(\bar{A}_m^\varepsilon, B_{m,j+1}, C_{m,j+1})$$

has a stabilizable and detectable eigenvalue on the unit disc. Let V be such that

$$VV' = I \text{ and } \ker V = \ker\langle C_{m,j+1} | \bar{A}_m^\varepsilon \rangle.$$

Since we might not be able to find a stable observer for the state x_m we actually construct an observer for the observable part of the state Vx_m. Because our triplet has a stabilizable and detectable eigenvalue on the unit disc, the observable part of the state Vx_m much contain at least one eigenvalue on the unit circle that can be stabilized. This motivates the following decentralized feedback law:

$$\begin{aligned}
v_i^m(k) &= K_i^\varepsilon x_m(k) + v_i^{m+1}(k), \qquad i = 1, \cdots, j, \\
p(k+1) &= A_s^\varepsilon p(k) + VB_{m,j+1} v_{j+1}^m(k) + K(C_{m,j+1}V'p(k) - y_{j+1}(k)) \\
v_{j+1}^m(k) &= F_\rho p(k) + v_{j+1}^{m+1}(k) \\
v_i^m(k) &= v_i^{m+1}(k), \qquad i = j+2, \cdots, \nu.
\end{aligned}$$

Here $p \in \mathbb{R}^s$ and A_s^ε is such that $A_s^\varepsilon V = V\bar{A}_m^\varepsilon$ and K is chosen such that $A_s^\varepsilon + KC_{m,j+1}V'$ has all its eigenvalues in the open unit disc and does not have any eigenvalues in common with $\bar{A}_k^\varepsilon$. Furthermore F_ρ is chosen in a way that $\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V$ has at least one less eigenvalue on the unit disc than A_m^ε and still all of its eigenvalues are in the closed unit disc and also $F_\rho \to 0$ as $\rho \to 0$. Defining

$$\bar{x}_{m+1} = \begin{pmatrix} x_m \\ p - Vx_m \end{pmatrix},$$

we have

$$\bar{x}_{m+1}(k+1) = \begin{pmatrix} \bar{A}_m^{\varepsilon} + B_{m,j+1}F_\rho V & B_{m,j+1}F_\rho \\ 0 & A_s^{\varepsilon} + KC_{m,j+1}V' \end{pmatrix} \bar{x}_{m+1}(k) + \sum_{i=1}^{\nu} \bar{B}_{m+1,i} v_i^{m+1}(k) \tag{3.6}$$

$$y_i(k) = \bar{C}_{m+1,i}\bar{x}_{m+1}(k) \quad i = 1, \cdots, \nu.$$

where

$$\bar{B}_{m+1,i} = \begin{pmatrix} B_{m,i} \\ -VB_{m,i} \end{pmatrix}, \quad \bar{C}_{m+1,i} = \begin{pmatrix} C_{m,i} & 0 \end{pmatrix}$$

for $i \neq j+1$ and

$$\bar{B}_{m+1,j+1} = \begin{pmatrix} B_{m,j+1} \\ 0 \end{pmatrix}, \quad \bar{C}_{m+1,j+1} = \begin{pmatrix} C_{m,j+1} & 0 \\ V & I \end{pmatrix}$$

It is easy to check that the above feedback laws satisfy the properties (1) and (2). What remains is to show that they satisfy property (3). Also we need to show that the control laws can be written in the form mentioned in step m for step $m+1$.

For any ε there exists a $R_m^{\varepsilon} > 0$ with

$$(A_s^{\varepsilon} + KC_{m,j+1}V')'R_m^{\varepsilon}(A_s^{\varepsilon} + KC_{m,j+1}V') - R_m^{\varepsilon} < 0$$

such that $R_m^{\varepsilon} \to 0$ as $\varepsilon \to 0$. Because $F_\rho \to 0$ as $\rho \to 0$, for each ε, for small enough ρ we have

$$\|F_\rho e\| < \frac{1}{2n} \qquad \forall\, e \text{ such that } e'R_m^{\varepsilon}e \leq n - m + \tfrac{1}{2}.$$

Note that $\bar{A}_m^{\varepsilon} + B_{m,j+1}F_\rho V$ has at least one less eigenvalue on the unit disc than $\bar{A}_m^{\varepsilon}$ and has all its eigenvalues in the close unit disc. Applying Lemma (3.5), for small ρ we have

$$(\bar{A}_m^{\varepsilon} + B_{m,j+1}F_\rho V)'\bar{P}_\rho^{\varepsilon}(\bar{A}_m^{\varepsilon} + B_{m,j+1}F_\rho V) - \bar{P}_\rho^{\varepsilon} \leq 0$$

with $\bar{P}_\rho^{\varepsilon} \to \bar{P}_m^{\varepsilon}$ as $\rho \to 0$.

Now note that $\bar{A}_m^\varepsilon$ and $A_s^\varepsilon + KC_{m,j+1}V'$ have disjoint eigenvalues we find that for small ρ, the matrices $\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V$ and $A_s^\varepsilon + KC_{m,j+1}V'$ have disjoint eigenvalues since $F_\rho \to 0$ as $\rho \to 0$. But then there exists a $W_{\varepsilon,\rho}$ such that

$$B_{m,j+1}F_\rho + (\bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V)W_{\varepsilon,\rho} - W_{\varepsilon,\rho}(A_s^\varepsilon + KC_{m,j+1}V') = 0$$

while $W_{\varepsilon,\rho} \to 0$ as $\rho \to 0$. Now we define $\bar{P}_{m+1}^{\varepsilon,\rho}$ to be

$$\bar{P}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} I & 0 \\ -W'_{\varepsilon,\rho} & I \end{pmatrix} \begin{pmatrix} \bar{P}_\rho^\varepsilon & 0 \\ 0 & R_m^\varepsilon \end{pmatrix} \begin{pmatrix} I & -W_{\varepsilon,\rho} \\ 0 & I \end{pmatrix}$$

We define

$$\bar{A}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{A}_m^\varepsilon + B_{m,j+1}F_\rho V & B_{m,j+1}F_\rho \\ 0 & A_s^\varepsilon + KC_{m,j+1}V' \end{pmatrix}$$

We will have the following properties

$$(\bar{A}_{m+1}^{\varepsilon,\rho})'\bar{P}_{m+1}^{\varepsilon,\rho}\bar{A}_{m+1}^{\varepsilon,\rho} - \bar{P}_{m+1}^{\varepsilon,\rho} \leq 0$$

and

$$\lim_{\rho\to 0} \bar{P}_{m+1}^{\varepsilon,\rho} = \begin{pmatrix} \bar{P}_m^\varepsilon & 0 \\ 0 & R_m^\varepsilon \end{pmatrix}$$

Now consider $\bar{x}_{m+1}$ such that

$$\bar{x}'_{m+1}\bar{P}_{m+1}^{\varepsilon,\rho}\bar{x}_{m+1} \leq n - m$$

Then with a small enough choice for ρ we have

$$x'_m\bar{P}_m^\varepsilon x_m \leq n - m + \frac{1}{2} \text{ and } (p - Vx_m)'R_m^\varepsilon(p - Vx_m) \leq n - m + \frac{1}{2} \tag{3.7}$$

Next for each ε we choose $\rho = \rho_\varepsilon$ such that (3.7) holds and we have

$$\|F_\rho V x_m\| < \frac{1}{2n} \qquad \forall\, x_m \text{ such that } x'_m\bar{P}_m^\varepsilon x_m \leq n - m + \frac{1}{2}$$

Next we must check the bounds on the inputs in step $m+1$. For $i = 1, \ldots, j$, we have

$$\|u_i\| = \|F^m_{i,\varepsilon} x_m + K^\varepsilon_i x_m\| \leq \frac{m}{n} + \frac{1}{2n} \leq \frac{m+1}{n}$$

For $i = j+1$, we have:

$$\|u_i\| = \|F^m_{i,\varepsilon} x_m + F_{\rho_\varepsilon} p\| = \|F^m_{i,\varepsilon} x_m + F_{\rho_\varepsilon} V x_m + F_{\rho_\varepsilon}(p - V x_m)\| \leq \frac{m}{n} + \frac{1}{2n} + \frac{1}{2n} = \frac{m+1}{n}$$

Finally, for $i = j+2, \ldots, \nu$, we have:

$$\|u_i\| = \|F^m_{i,\varepsilon} x_m\| \leq \frac{m}{n} \leq \frac{m+1}{n}$$

Now for $i \neq j+1$ we set $n_{i,m+1} = n_{i,m}$ and for $i = j+1$ we set $n_{i,m+1} = n_{i,m} + s$.

If $n_{i,m} > 0$ we choose

$$p^{m+1}_i = \begin{pmatrix} p^m_i \\ \\ p \end{pmatrix}$$

and if $n_{i,m} = 0$ we choose $p^{m+1}_i = p$. Now we are able to express the system in terms of x_{m+1}. We introduce a basis transformation T_{m+1} such that $\bar{x}_{m+1} = T_{m+1} x_{m+1}$. Next, we define

$$P^\varepsilon_{m+1} = T'_{m+1} \bar{P}^{\varepsilon,\rho_\varepsilon}_{m+1} T_{m+1}.$$

Now for $i = 1, \ldots, \nu$ depending on the value of $n_{i,m+1}$ we can rewrite the control laws in the desired form and subsequently the properties (1), (2) and (3) are obtained.

Now we know that there exists a value of m, say $\ell \leq n$, such that A^ε_ℓ has all its eigenvalues in the open unit disc. We set $\nu^\ell_i = 0$ for $i = 1, 2, 3, \ldots, \nu$. Then the decentralized control laws $\Sigma^{\ell,\varepsilon}_i$, $i = 1, 2, 3, \ldots, \nu$ together represent a decentralized semi global feedback law for the system Σ. In other words, we claim that for any given compact sets $W \subset \mathbb{R}^n$ and $S_i \subset \mathbb{R}^{n_{i,\ell}}$ for $i = 1, 2, 3, \ldots, \nu$, there exists an ε^* such that the origin of the closed loop system is exponentially stable for any $0 < \varepsilon < \varepsilon^*$ and the compact set $W \times S_1 \times \cdots S_\nu$ is within the domain of attraction. Furthermore for all initial conditions within $W \times S_1 \times \cdots S_\nu$, the closed loop system behaves like a linear system, that is the saturation is not activated.

We know that for ε small enough, the set

$$\Omega_1^\varepsilon := \left\{ x_\ell \in \mathbb{R}^{n_\ell} \mid x_\ell' P_\ell^\varepsilon x_\ell \leq 1 \right\}$$

is inside the domain of attraction of the equilibrium point of the closed loop system comprising the given system Σ and the decentralized control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, 2, 3, \ldots, \nu$ because for all initial conditions within Ω_1^ε, it is obvious that $\|u_i\| \leq 1$, $i = 1, 2, 3, \ldots, \nu$ which means that the closed loop system behaves like a linear system, that is the saturation is not activated. Furthermore since all of the eigenvalues of A_ℓ^ε are in the open unit disc, this linear system is asymptotically stable. In addition because of the fact that $P_\ell^\varepsilon \to 0$ as $\varepsilon \to 0$, we find that $W \times S_1 \times \cdots S_\nu$ is inside Ω_1^ε for ε sufficiently small. This concludes that the decentralized control laws $\Sigma_i^{\ell,\varepsilon}$, $i = 1, 2, 3, \ldots, \nu$ are semi-globally stabilizing.

4. TIME VARYING CONTROLLERS IN DISCRETE-TIME DECENTRALIZED CONTROL

In this chapter, we consider the problem of finding a time-varying controller which can stabilize a decentralized discrete-time system [18]. In continuous-time, it was already known that time-varying decentralized controllers can achieve stabilization in cases where time-invariant decentralized controllers cannnot. This paper works out the details for the discrete-time case.

4.1 Introduction

The result presented here contributes to our ongoing study of the stabilization of decentralized systems. The eventual goal of this study is the *design* of controllers for decentralized systems that achieve not only stabilization but also high performance. As a first step toward this design goal, we are currently looking for tight conditions on a decentralized plant for the *existence* of stabilizing controllers.

To motivate and introduce the main result in the article, let us briefly review foundational studies on both decentralized control and saturating control systems. We recall that a necessary and sufficient condition for stabilization of a decentralized system using linear, time-invariant state-space controllers is given in Wang and Davison's classical work [22]. They obtain that stabilization is possible if and only if all *decentralized fixed modes* of a plant are in the open left-half plane, and give specifications of and methods for finding these decentralized fixed modes. Numerous further characterizations of decentralized stabilization (and fixed modes) have been given, see for instance the work of Corfmat and Morse [14]. For time-varying controllers, the result for continuous-time systems goes back to the paper [17]. In this paper, it was established that there are systems for which there do not exist time-invariant, linear decentralized controllers while there do exist time-varying, linear decentralized controllers. Moreover, they obtain that stabilization is possible if and only if all *quotient fixed modes* of a plant are in the open left-half plane. For nonlinear, time-invariant systems some first results are presented in [21]. It has been shown in [20] that nonlinear, time-varying decentralized controllers can only stabilize a system when it can also be stabilized by a linear, time-varying decentralized controller.

In this chapter we focus on time-varying, decentralized controllers for discrete-time systems. The continuous-time results in [17, 20] show that the existence results are related to the so-called quotient fixed modes while for time-invariant controllers the existence is related to decentralized fixed modes. Not surprisingly, in this case stabilization by time-invariant controllers is possible if the decentralized fixed modes are in the open unit disc while for time-varying controllers stabilization is possible if the quotient fixed modes are in the open unit disc

In this paper, we first present our main results and then we present in Section 4.3, a detailled outline of the proof of the crucial underlying lemma.

4.2 Main Result

Consider a decentralized control system Σ:

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{\nu} B_i u_i(k) \\ \\ y_i(k) \quad = C_i x(k), \qquad i = 1, \ldots, \nu, \end{cases} \tag{4.1}$$

As in many control systems the first objective is to find a controller which can stabilize the given system under the presented decentralized structure. In other words, when do there exists ν controllers of the form:

$$\Sigma_i : \begin{cases} z_i(k+1) = K_i z_i(k) + L_i y_i(k), \quad z_i \in \mathbb{R}^{s_i} \\ \\ u_i(k+1) = M_i z_i(k) + N_i y_i(k). \end{cases} \tag{4.2}$$

such that the interconnection of (4.1) and (4.2) is asymptotically stable. From the paper [22] it is known for continuous-time systems that this is related to so-called decentralized fixed modes:

Definition 4.1. *Consider the system Σ. $\lambda \in \mathbb{C}$ is called a decentralized* ***fixed mode*** *if for all matrices $H_1, \ldots H_\nu$ of appropriate dimension we have*

$$\det\left(\lambda I - A - \sum_{i=1}^{\nu} B_i H_i C_i\right) = 0$$

The following result goes back to [14, 22] for the continuous-time. The discrete-time equivalent follows quite directly and can for instance be found explicitly in [19].

Theorem 4.2. *A necessary and sufficient condition for the existence of a decentralized feedback control law of the form* (4.2) *for the system Σ such that the closed loop system is asymptotically stable is that all the fixed modes of the system are asymptotically stable (i.e. in the unit disc).*

In case the above condition is not satisfied a natural question is to check whether we can find a suitable controller if we expand the class of controllers to time-varying or nonlinear systems.

For time-varying controllers, again for continuous-time systems it is known that the result is related to the so-called quotient fixed modes. This result was initially presented in [17] while later further insight was obtained in the paper [20] . Let is first formally define quotient fixed modes:

Definition 4.3. *Consider the system* (4.1)*. We define a directed graph with nodes* $\{1, \ldots, \nu\}$ *with an edge from node* i *to node* j *if there exists an integer* k *such that* $C_j A^k B_i \neq 0$*. Let* ν^* *be the number of strongly connected components of this directed graph. Let* u_j^* *be the union of all inputs associated to the* j*'th strongly connected component, i.e.* u_i *is part of* u_j^* *if and only if node* i *is part of the* j*th strongly connected component of the graph. Similarly let* y_j^* *be the union of all outputs associated to the* j*'th strongly connected component.*

Without loss of generality we assume that the nodes are ordered in such a way that strongly connected component j *consists of nodes* $\{i_{j-1} + 1, \ldots, i_j\}$ *with* $i_0 = 0$ *an* $i_{\nu^*} = \nu$*. Moreover, strongly connected component* j_1 *can connect to strongly connected component* j_2 *only if* $j_2 \leq j_1$*.*

Using the above, the matrices in the system equation will, in an appropriate basis have the *following triangular structure*

$$A = \begin{pmatrix} \tilde{A}_1 & \times & \cdots & \times \\ 0 & \tilde{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \times \\ 0 & \cdots & 0 & \tilde{A}_{\nu^*} \end{pmatrix},$$

$$B = \begin{pmatrix} B_1^* & B_2^* & \cdots & B_{\nu^*}^* \end{pmatrix} = \begin{pmatrix} \tilde{B}_1 & \times & \cdots & \times \\ 0 & \tilde{B}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \times \\ 0 & \cdots & 0 & \tilde{B}_{\nu^*} \end{pmatrix},$$

$$C = \begin{pmatrix} C_1^* \\ C_2^* \\ \vdots \\ C_{\nu^*}^* \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 & \times & \cdots & \times \\ 0 & \tilde{C}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \times \\ 0 & \cdots & 0 & \tilde{C}_{\nu^*} \end{pmatrix},$$

where

$$B_j^* = \begin{pmatrix} B_{i_{j-1}+1} & \cdots B_{i_j-1} & B_{i_j} \end{pmatrix}$$

$$C_j^* = \begin{pmatrix} C_{i_{j-1}+1} \\ \vdots \\ C_{i_j-1} \\ C_{i_j} \end{pmatrix}$$

and

$$\tilde{B}_j^* = \begin{pmatrix} \tilde{B}_{i_{j-1}+1} & \cdots \tilde{B}_{i_j-1} & \tilde{B}_{i_j} \end{pmatrix}$$

$$\tilde{C}_j^* = \begin{pmatrix} \tilde{C}_{i_{j-1}+1} \\ \vdots \\ \tilde{C}_{i_j-1} \\ \tilde{C}_{i_j} \end{pmatrix}$$

for $j = 1, \ldots, \nu^*$.

We call $\lambda \in \mathbb{C}$ *a* ***quotient fixed mode*** *if* λ *is a (centralized) uncontrollable or unobservable eigenvalue of* $(\tilde{A}_j, \tilde{B}_j, \tilde{C}_j)$ *for some* $j \in \{1, \ldots, \nu^*\}$

It is not difficult to verify, using the upper-diagonal structure, that if we find stabilizing controllers for the subsystems:

$$\Sigma^j : \begin{cases} x_j(k+1) = \tilde{A}_j x(k) + \sum_{i=i_{j-1}+1}^{i_j} \tilde{B}_i u_i^*(k) \\ \quad y_i(k) \quad = \tilde{C}_i x(k), \qquad i = i_{j-1}, \ldots, i_j, \end{cases} \tag{4.3}$$

for $j = 1, \ldots, \nu^*$ then if we combine these controllers we find a stabilizing controller for the overall system. Conversely, we can only find a decentralized controller for the overall system if we can find decentralized controllers for each of the subsystems of the form (4.3). If all quotient fixed modes are inside the unit circle then these subsystems are all (centrally) stabilizable and detectable. Moreover, the graph associated with these subsystems is strongly connected.

The claim is that a decentralized system which is centrally stabilizable and detectable and whose associated graph is strongly connected can always be stabilized in a decentralized manner if we allow for time-varying controllers. This would immediately yield that the overall system is stabilizable by a decentralized time-varying controller if and only if all quotient fixed modes are inside the unit circle. The above claim which is crucial for this paper is formulated in the following lemma:

Lemma 4.4. *Consider a system* Σ *of the form* (4.1) *which is centrally stabilizable and detectable and which is such that the associated directed graph is strongly connected. Then there exist* ν^* *linear, time-varying controllers of the form* (4.4) *such that the resulting closed loop system is asymptotically stable.*

The above lemma immediately yields the main result of this paper:

Theorem 4.5. *Consider a system* Σ *of the form* (4.1) *for which all quotient fixed modes are inside the unit circle. Then there exist* ν^* *linear, time-varying controllers of the form:*

$$\Sigma_i : \begin{cases} z_i(k+1) = K_i(k)z_i(k) + L_i(k)y_i(k), \quad z_i \in \mathbb{R}^{s_i} \\ u_i(k+1) = M_i(k)z_i(k) + N_i(k)y_i(k). \end{cases} \tag{4.4}$$

such that the resulting closed loop system is asymptotically stable.

It is easily verified that quotient fixed modes are a subset of the decentralized fixed modes which makes the above theorem consistent with Theorem 4.2. However, there are examples of systems which can be stabilized via decentralized time-varying controllers but which can not be stabilized via decentralized time-invariant controllers.

4.3 Proof of Lemma 4.4

Consider a system Σ of the form (4.1) which is centrally stabilizable and detectable and which is such that the associated directed graph is strongly connected. We will first prove their exists a static decentralized and time-varying (but periodic) preliminary feedback such that the system is uniformly detectable from channel 1. Let x be such that

$$\begin{pmatrix} C_1 \\ C_1 A \\ \vdots \\ C_1 A^{n-1} \end{pmatrix} x = 0$$

Next, consider an initial condition $x(0) = x$ such that $y(j) = 0$ for $j = 0, \ldots, n-1$. Assume $x(n) \neq 0$ and $x(n)$ is observable through another channel j, i.e. $C_j A^s x(n) \neq 0$ for some $s \geq 0$. Note that this implies $C_j A^p x \neq 0$ for $p = s + n$. We will prove that there exists a preliminary feedback such that $y_1(\tilde{p}) \neq 0$ for some $\tilde{p} > n$.

Consider the associated strongly connected directed graph. Let $\{j, j_v, \ldots, j_1, 1\}$ be the shortest path from node j to node 1. By definition, this implies there exists $p_1, \ldots, p_{v+1}$ such that:

$C_1 A^{p_1} B_{j_1} \neq 0 \;\; C_{j_1} A^{p_2} B_{j_2} \neq 0 \;\; \cdots C_{j_{v-1}} A^{p_v} B_{j_v} \neq 0 \;\; C_{j_v} A^{p_{v+1}} B_j \neq 0$

Then there exists $K_1, \ldots, K_{v+1}$ such that

$C_1 A^{p_1} B_{j_1} K_1 C_{j_1} A^{p_2} B_{j_2} K_2 \times \cdots \times C_{j_{v-1}} A^{p_v} B_{j_v} K_v C_{j_v} A^{p_{v+1}} B_j K_{v+1} C_j A^p x \neq 0$

From the fact that we have chosen a shortest path it is easily verified that

$C_{j_{\ell-1}} A^{p_\ell} B_{j_\ell} K_{j_\ell} C_{j_\ell} A^{p_{\ell+1}} B_{\ell+1} = C_{j_{\ell-1}} A^{p_\ell} \left(A + B_{j_\ell} K_{j_\ell} C_{j_\ell}\right) A^{p_{\ell+1}} B_{j_{\ell+1}}$

for any $\ell \in \{1, \ldots, v\}$ (with $j_0 = 1$). If this would not be true then

$$C_{j_{\ell-1}} A^{p_\ell} A A^{p_{\ell+1}} B_{j_{\ell+1}} \neq 0$$

which would imply an edge from $j_{\ell+1}$ to $j_{\ell-1}$ and a shorter path from j to 1. Choose the preliminary feedback:

$$
\begin{aligned}
u_j(p) &= \varepsilon K_{v+1} y_j(p) + v_j(p) \\
u_{j_v}(p + p_v + 1) &= \varepsilon K_v y_{j_v}(p + p_v + 1) \\
&\quad + v_{j_v}(p + p_v + 1) \\
u_{j_{v-1}}(p + p_v + p_{v-1} + 2) &= \varepsilon K_{v-1} y_{j_{v-1}}(p + p_v + p_{v-1} + 2) \\
&\quad + v_{j_{v-1}}(p + p_v + p_{v-1} + 2) \\
\vdots \;\; & \vdots \\
u_{j_1}(p + p_v + \cdots + p_1 + v) &= \varepsilon K_1 y_{j_1}(p + p_v + \cdots + p_1 + v) \\
&\quad + v_{j_1}(p + p_v + \cdots + p_1 + v)
\end{aligned}
$$

while for all other channels and all other time instances we have no preliminary feedback. After this preliminary feedback we obtain a system:

$$
\Sigma : \begin{cases} x(k+1) = A_\varepsilon(k)x(k) + \sum_{i=1}^{\nu} B_i v_i(k) \\ \quad y_i(k) \quad = C_i x(k), \qquad i = 1, \ldots, \nu, \end{cases} \tag{4.5}
$$

for $k = 1, \ldots, \tilde{p}$ where

$$
\tilde{p} = p + p_v + \ldots + p_1 + v
$$

In that case $x(0) = x$ yields

$$
y_1(\tilde{p}) \neq 0
$$

for all $\varepsilon > 0$ given $v_i(k) = 0$ for $k = 1, \ldots, \tilde{p}$ and $i = 1, \ldots, \nu$. Next, consider an initial condition $\tilde{x}$ such that $y_1(k) = 0$ for $k = 1, \ldots, \tilde{p}$ while there exists a channel $\tilde{j}$ such that $C_{\tilde{j}} A^s \tilde{x} \neq 0$ for some

$s > \tilde{p}$. This clearly implies that

$$A^{\tilde{p}}\tilde{x} \neq 0$$

but then

$$\bar{x}_\varepsilon = \prod_{k=1}^{\tilde{p}} A_\varepsilon(k)\tilde{x} \to A^{\tilde{p}}\tilde{x}$$

as $\varepsilon \to 0$ and hence for ε small enough we have

$$\bar{x}_\varepsilon \neq 0 \qquad C_{\tilde{j}} A^{s-\tilde{p}} \bar{x}_\varepsilon \neq 0$$

Then, as before, we can find a preliminary feedback for $k = \tilde{p}, \ldots, \bar{p}$ such that we obtain a system (4.5) with the property that for $x(0) = \tilde{x}$ we obtain:

$$y_1(\bar{p}) \neq 0$$

given $v_i(k) = 0$ for $k = 1, \ldots, \bar{p}$ and $i = 1, \ldots, \nu$. We can repeat this algorithm again if we can find an $\hat{x}$ such that $y_1(k) = 0$ for $k = 1, \ldots, \bar{p}$ given $v_i(k) = 0$ for $k = 1, \ldots, \bar{p}$ and $i = 1, \ldots, \nu$ while

$$C_j A^k \hat{x} \neq 0$$

for some $k > \bar{p}$. However, because of the linearity of the system, the algorithm will end after at most n steps because of the dimension of the state space. After the algorithm is completed we find that there exists $p > 0$ such that for any initial condition x_0 either it is observable through y_1 on the time interval $[0, p]$ or

$$C_j A^s x_0$$

is equal to zero for all $j = 1, \ldots, \nu$ and for all $s > p$. This implies that $A^p x_0$ is (centrally) unobservable for the original system, i.e.

$$x_0 \in \ker A^p + \langle \ker C, A \rangle$$

where

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_\nu \end{pmatrix}$$

It is then easily verified that $x(k) = A^k x_0$ given $v_i(k) = 0$ for $k = 1, \ldots, \bar{p}$ and $i = 1, \ldots, \nu$. Since, by assumption (C, A) was detectable we find that $A^k x_0 \to 0$ as $k \to \infty$. If we extend this preliminary feedback periodically it is easily verified that we obtain a system which is uniformly detectable through channel 1.

However, we also need stabilizability. We have chosen a preliminary feedback for $k = 1, \ldots, p$ to achieve detectability. Next, we can, through a dual argument, choose a preliminary feedback for $k = p + 1, \ldots, q$ such that the system becomes stabilizable. Clearly, this need not be done succesively in time but it makes the argument a lot easier. We simply look at $x(p)$ and check whether we can make those states stabilizable through channel 1 on the interval $[p + 1, q]$. This argument is completely dual to the design to make the system detectable through channel 1 on the interval $[0, p]$.

In this way, we obtain a preliminary feedback such that we have detectability on the interval $[0, p]$ and detectability on the interval $[p + 1, q]$. By making this preliminary feedback periodic, we obtain a system which is uniformly stabilizable and detectable through channel 1. It is well known that this implies that there exists a stabilizing, dynamic controller for channel 1. The latter completes the proof of Theorem 4.4.

4.4 Conclusion

This chapter presented a necessary and sufficient conditions for the existence of stabilizing time-varying linear controllers for a discrete-time decentralized system. Like the earlier continuous-time results, the proof of the existence of a suitable controller is constructive but is certainly not efficient enough to generate an easily implementable controller. For decentralized system, there is still a strong need for suitable design methodologies which is a main topic of our future research. The lack of a suitable design methodology for general decentralized systems is not limited to time-varying designs. Also if all decentralized fixed modes are asymptotically stable, then we still do not have a suitable design for a general decentralized system. For certain systems with special additional structure such designs of course exist.

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